

**Character correspondences
above fully ramified sections
and Schur indices**

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ABSTRACT. Let N be a finite group of odd order and A a finite group that acts on N such that $|N|$ and $|A|$ are coprime. Isaacs constructed a natural correspondence between the set $\text{Irr}_A(N)$ of irreducible complex characters invariant under the action of A , and the set $\text{Irr}(\mathbf{C}_N(A))$. We show that this correspondence preserves Schur indices over the rational numbers \mathbb{Q} . Moreover, suppose that the semidirect product AN is a normal subgroup of the finite group G and set $U = \mathbf{N}_G(A)$. Let $\chi \in \text{Irr}_A(N)$ and $\chi^* \in \text{Irr}(\mathbf{C}_N(A))$ correspond. Then there is a canonical bijection between $\text{Irr}(G \mid \chi)$ and $\text{Irr}(U \mid \chi^*)$ preserving Schur indices. We also give simplified and more conceptual proofs of (known) character correspondences above fully ramified sections.

1. INTRODUCTION

Let G be a finite group and let $L \subseteq K$ be normal subgroups of G . Suppose $\varphi \in \text{Irr } L$ is fully ramified in K . This means that φ is invariant in K and that there is a unique irreducible character $\vartheta \in \text{Irr } K$ lying above φ . This situation occurs naturally in the character theory of finite solvable groups, and a number of authors has studied this situation [4, 8, 10, 11, 19, 20]. Under additional conditions, there is a subgroup $H \leq G$ with $G = KH$ and $K \cap H = L$ (see Figure 1), and a correspondence between $\text{Irr}(G \mid \varphi)$ and $\text{Irr}(H \mid \vartheta)$. In particular, Isaacs [8] constructs such a bijection,

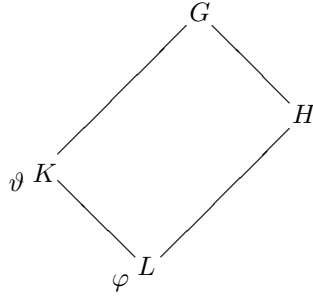


FIGURE 1.

when K/L is abelian of odd order. He shows that there is a canonical character ψ defined on H/L , all of whose values are nonzero, and that the equation $\chi_H = \psi\xi$ does define a bijection between $\chi \in \text{Irr}(G \mid \varphi)$ and $\xi \in \text{Irr}(H \mid \vartheta)$. The construction of the character ψ is rather lengthy and intricate.

In this paper, we show that the results of Isaacs can be deduced from our theory of “magic representations” [14, 15]. In fact, this theory arose from an attempt to better understand the correspondence of Isaacs. The idea is as follows: Suppose φ is

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invariant in G . Let e_φ be the central primitive idempotent of $\mathbb{C}L$ associated with φ . Since φ is fully ramified in K , we have $e_\varphi = e_\vartheta$, where $\{\vartheta\} = \text{Irr}(K \mid \varphi)$. (In fact, this is equivalent to φ being fully ramified in K .) Set $S = (\mathbb{C}Ke_\varphi)^L = \mathbb{C}_{\mathbb{C}Ke_\varphi}(L)$. Then $S \cong \mathbf{M}_n(\mathbb{C})$. The factor group G/L acts on S . Since all automorphisms of $S \cong \mathbf{M}_n(\mathbb{C})$ are inner automorphisms, there is $\sigma(x) \in S^*$ for each $x \in G/L$ such that $s^x = s^{\sigma(x)}$ for all $s \in S$. This yields a projective representation $\sigma: G/L \rightarrow S$. If we can choose the $\sigma(x)$ such that the restriction of σ to H/L is an ordinary group representation, then we call $\sigma: H/L \rightarrow S$ a *magic representation*. It is fairly easy to show that $\mathbb{C}Ge_\varphi \cong \mathbf{M}_n(\mathbb{C}He_\varphi)$ when a magic representation exists. This explains the existence of a character correspondence. If ψ is the character of σ , then $\chi_H = \psi\xi$ for corresponding $\chi \in \text{Irr}(G \mid \varphi)$ and $\xi \in \text{Irr}(H \mid \varphi)$.

These results apply to character fives in general. They are explained in Section 4. (Sections 2–3 contain preliminary material.) Section 5 contains results about magic representations for character fives. In Section 6, we give a very short and easy proof of a result including some results of Lewis [19, 20]. In Section 7, we show that there is a magic representation when K/L is abelian of odd order. In Section 8, we show that there is a canonical choice for the magic representation, thereby proving the existence of a canonical bijection. These two sections yield a new proof of Isaacs' result [8].

The approach described so far works in fact for smaller fields than \mathbb{C} , but the field has to contain the values of φ . In a second part of the paper, we drop the assumption that the field contains the values of φ . We also drop the assumption that φ is invariant in G . There is a unique central primitive idempotent, f , in $\mathbb{Q}L$, such that $\varphi(f) \neq 0$. Using Clifford theory, one sees that it is no loss of generality to assume that f is invariant in G . This means that the Galois orbit of φ is invariant in G , but φ itself may not be invariant. We are able to construct an explicit isomorphism $\mathbb{Q}Gf \cong \mathbf{M}_n(\mathbb{Q}Hf)$, when K/L is abelian of odd order, and an additional condition is given (Theorem 10.3). The proof of this result, which occupies Sections 11 and 13, may be considered as the heart of this paper. The proof relies on the approach using magic representations.

The assumption that φ is fully ramified in K may be skipped. The more general result follows from Theorem 10.3 by reduction arguments that are more or less standard. (However, the “going down” theorem for semi-invariant characters, Proposition 14.2, might be new.)

Isaacs [8] gave two applications of his study of fully ramified sections. The first is now known as the Isaacs part of the Glauberman-Isaacs correspondence: Suppose a group, A , acts on another group, N , such that $|A|$ and $|N|$ are relatively prime. In case $|N|$ is odd, Isaacs constructed a natural correspondence between $\text{Irr}_A(N)$, the set of irreducible characters of N invariant under the action of A , and $\text{Irr } C_N(A)$. As an application of our results, we get that this correspondence preserves Schur indices over all fields. (This is wrong for the Glauberman correspondence, as the example of the quaternion group with a C_3 acting on it shows.) Even more is true: Suppose that the semidirect product, AN , is an invariant subgroup of some finite group G . Set $U = \mathbf{N}_G(A)$ and $C = \mathbf{C}_N(A) = N \cap U$. Let $\chi \in \text{Irr}_A(N)$ and let $\chi^* \in \text{Irr } C$ be its Isaacs correspondent. There is a unique primitive idempotent i in $(\mathbb{Q}N)^G = \mathbf{C}_{\mathbb{Q}N}(G)$ such that $\chi(i) \neq 0$, and a similar defined idempotent i^* in $(\mathbb{Q}C)^U$. Then $\mathbb{Q}Gi \cong \mathbf{M}_n(\mathbb{Q}Ui^*)$, with $n = \chi(1)/\chi^*(1)$, and there is a canonical correspondence between $\text{Irr}(G \mid \chi)$ and $\text{Irr}(U \mid \chi^*)$.

In his second application, Isaacs constructed, for a group G of odd order, a bijection between the set of irreducible characters of G with degree not divisible by a given prime p , and the set of such characters of the normalizer of a Sylow p -subgroup, thereby proving the McKay conjecture for groups of odd order. Let us

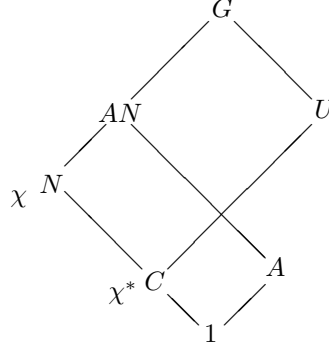


FIGURE 2. Above the Isaacs correspondence

mention that Turull [26, 28] showed that this bijection preserves Schur indices over any field, if $|G|$ is odd. His arguments are, however, quite different from those given here.

2. GOOD ELEMENTS

We review the concept of *good elements* introduced by Gallagher [6, p. 177]. It is related to a bilinear form introduced by Isaacs [8] (Isaacs attributes the form to Dade). Let $L \trianglelefteq G$. Suppose that $\varphi \in \text{Irr } L$ is invariant in G and let \mathbb{F} be a field containing the values of φ . Let e_φ be the central primitive idempotent in $\mathbb{F}L$ associated with φ . Then any $g \in G$ acts on $\mathbb{F}Le_\varphi$ by conjugation. Since $\mathbb{F}Le_\varphi$ is central simple, by the Skolem-Noether theorem there is $c_g \in (\mathbb{F}Le_\varphi)^*$ such that $a^g = a^{c_g}$ for all $a \in \mathbb{F}Le_\varphi$. The element c_g is determined up to multiplication with elements of \mathbb{F} by this property.

If $x, y \in G$ with $[x, y] \in L$ then $[x, y]e_\varphi$ and $[c_x, c_y]$ induce the same action (by conjugation) on $\mathbb{F}Le_\varphi$ and so these elements differ by some scalar. We denote this scalar by $\langle\langle x, y \rangle\rangle_\varphi \in \mathbb{F}$. So by definition,

$$\langle\langle x, y \rangle\rangle_\varphi e_\varphi = [x, y][c_y, c_x].$$

This definition is independent of the choice of c_x and c_y , since this choice is unique up to multiplication with scalars.

Alternatively, assume that φ is afforded by a representation $\rho: L \rightarrow \mathbf{M}_{\varphi(1)}(\mathbb{F})$. For $g \in G$ there is $\gamma_g \in \mathbf{M}_{\varphi(1)}(\mathbb{F})$ with $\rho(l^g) = \rho(l)^{\gamma_g}$ for all $l \in L$. If $[x, y] \in L$, then $\rho([x, y])[\gamma_y, \gamma_x]$ centralizes $\rho(L)$, and thus it is a scalar matrix. Define $\langle\langle x, y \rangle\rangle_\varphi$ by $\langle\langle x, y \rangle\rangle_\varphi I = \rho([x, y])[\gamma_y, \gamma_x]$. Since the restriction of ρ to $\mathbb{F}Le_\varphi$ is an isomorphism between $\mathbb{F}Le_\varphi$ and $\mathbf{M}_{\varphi(1)}(\mathbb{F})$, both definitions agree. From the first definition we see, however, that $\langle\langle x, y \rangle\rangle_\varphi \in \mathbb{Q}(\varphi)$, while for the second we have to assume that φ is afforded by a representation over \mathbb{F} . On the other hand the second definition works for absolutely irreducible representations over fields of any characteristic.

Isaacs' definition [8, p. 596] is different, but from the definition given here it is easier to prove that $\langle\langle \cdot, \cdot \rangle\rangle_\varphi$ is indeed a bilinear alternating form. (I learned this definition from Knörr.)

In most of this work, φ will be fixed, and so we drop the index if no confusion can arise.

2.1. Lemma. *Let $g, x, x_1, x_2, y \in G$ with $[x, y], [x_i, y] \in L$ and $l_1, l_2 \in L$, and define $\langle\langle x, y \rangle\rangle_\varphi = \langle\langle x, y \rangle\rangle$ as above. Then*

- (a) $\langle\langle x_1 x_2, y \rangle\rangle = \langle\langle x_1, y \rangle\rangle \langle\langle x_2, y \rangle\rangle$.
- (b) $\langle\langle y, x \rangle\rangle = \langle\langle x, y \rangle\rangle^{-1}$.
- (c) $\langle\langle x l_1, y l_2 \rangle\rangle = \langle\langle x, y \rangle\rangle$.

$$(d) \llbracket x^g, y^g \rrbracket = \llbracket x, y \rrbracket.$$

Proof. All assertions can be verified with routine calculations using commutator identities. \square

In particular, $\llbracket \cdot, \cdot \rrbracket$ is constant on cosets of L , so we may view $\llbracket \cdot, \cdot \rrbracket$ as being defined on certain elements of $G/L \times G/L$, and we will do so whenever convenient.

Another trivial remark is this: Suppose that another group, A , acts on G (we use exponential notation g^a for the action) and stabilizes φ (that is, $L^a = L$ and $\varphi(l^a) = \varphi(l)$ for $a \in A$ and $l \in L$). Then if $x \in G$ and $a \in A$ with $[x, a] = x^{-1}x^a \in L$, the form $\llbracket x, a \rrbracket_\varphi$ is still defined. This is clear since we may work in the semidirect product AG , with the usual identifications of G , L and A with subgroups of the semidirect product. So we will sometimes use the notation $\llbracket x, a \rrbracket_\varphi$ in this more general situation without further explanation.

The definition of the form given by Isaacs [8, p. 596] was from the next lemma for $H = \langle L, h \rangle$. It shows that the form can be computed using only characters.

2.2. Lemma. *Let $L \leq H \leq G$ and χ be a classfunction of H with all its irreducible constituents lying over φ . Let $h \in H$ and $g \in G$ with $[h, g] \in L$. Then $\chi(h^g) = \chi(h)\llbracket h, g \rrbracket$.*

Proof. We work in the subgroup $\langle L, h \rangle$ of H . Writing $\chi_{\langle L, h \rangle}$ as a linear combination of irreducible characters lying above φ , we see that it is no loss to assume that $H = \langle L, h \rangle$, and that χ is irreducible. Since H/L is then cyclic and $\chi \in \text{Irr}(H \mid \varphi)$, in fact χ extends φ . Let $\hat{\rho}: H \rightarrow \mathbf{M}_{\varphi(1)}(\mathbb{C})$ be a representation affording χ that extends the representation ρ affording φ . Choose $\gamma_g \in \mathbf{M}_{\varphi(1)}(\mathbb{C})$ with $\rho(l^g) = \rho(l)^{\gamma_g}$, and let $\gamma_h = \hat{\rho}(h)$. Then

$$\begin{aligned} \hat{\rho}(h^g) &= \hat{\rho}(h[h, g]) = \hat{\rho}(h)\rho[h, g][\gamma_g, \gamma_h][\gamma_h, \gamma_g] \\ &= \hat{\rho}(h)\llbracket h, g \rrbracket[\hat{\rho}(h), \gamma_g] \\ &= \hat{\rho}(h)^{\gamma_g}\llbracket h, g \rrbracket. \end{aligned}$$

Taking the trace yields the desired result. \square

We continue to assume that $\varphi \in \text{Irr } L$ is invariant in G , where $L \trianglelefteq G$, and \mathbb{F} is a field containing the values of φ . We review some known material that yields other ways to compute the form $\llbracket \cdot, \cdot \rrbracket_\varphi$. Remember that $A = \mathbb{F}Ge_\varphi$ is naturally graded by the group G/L : For $x = Lg \in G/L$, set $A_x = \mathbb{F}Le_\varphi g$ (of course, this is independent of the choice of $g \in x$). Then $A_x A_y = A_{xy}$ and $A = \bigoplus_{x \in G/L} A_x$. Now let $S = \mathbf{C}_A(A_1) = \mathbf{C}_A(L) = (\mathbb{F}Ge_\varphi)^L$. The grading of A yields a grading of S , namely $S = \bigoplus_{x \in G/L} S_x$ with $S_x = \mathbf{C}_{A_x}(A_1)$.

A graded unit of S is a unit of S that is contained in some S_x . The set of all graded units of a graded algebra forms a group. It is well known that in the situation at hand, S_x contains units for all $x \in G/L$. Namely, for $g \in G$ there is $c_g \in \mathbb{F}Le_\varphi$ with $a^g = a^{c_g}$ for all $a \in \mathbb{F}Le_\varphi$, and then $s_g := c_g^{-1}g = gc_g^{-1} \in \mathbf{C}_{\mathbb{F}Le_\varphi g}(L) = S_{Lg}$. Let $g, h \in G$ with $[g, h] \in L$. Then

$$\begin{aligned} [s_g, s_h] &= [s_g, h] = [c_g^{-1}, h]^g [g, h] = [c_g^{-1}, c_h]^{c_g} [g, h] \\ &= [c_h, c_g][g, h] = \llbracket g, h \rrbracket_\varphi, \end{aligned}$$

where the first equality follows since s_g and $c_h \in A_1$ commute, and the third equality follows since c_g^{-1} and $[c_g^{-1}, h]$ are elements of $\mathbb{F}Le_\varphi$. Since $S_{Lg} = \mathbb{F}s_g$ and similarly for S_{Lh} , this is true for every choice of units $s_x \in S_{Lg}$, $s_y \in S_{Lh}$.

Suppose we choose a unit $s_x \in S_x$ for every $x \in G/L$. Then a cocycle $\alpha: G/L \rightarrow \mathbb{F}^*$ is defined by $s_x s_y = \alpha(x, y) s_{xy}$. We have

$$\langle\langle x, y \rangle\rangle_\varphi = [s_x, s_y] = \frac{\alpha(x, y)}{\alpha(y, x)}.$$

(Note that α depends on the choice of the s_x , but its cohomology class does not.) We have thus proved the following lemma:

2.3. Lemma. *Hold the notation just introduced, and let $x, y \in G/L$ with $[x, y] = 1_{G/L}$. Then*

$$\langle\langle x, y \rangle\rangle_\varphi e_\varphi = [s_x, s_y] = s_x^{-1} s_x^y \quad \text{and} \quad \langle\langle x, y \rangle\rangle_\varphi = \frac{\alpha(x, y)}{\alpha(y, x)}.$$

Note that the equation $\langle\langle x, y \rangle\rangle_\varphi e_\varphi = s_x^{-1} s_x^y$ is still true if y is an element of some group acting on G and stabilizing φ , and such that $x \in \mathbf{C}_{G/L}(y)$.

2.4. Definition. Let $L \leq H \leq G$ and $g \in G$ (or $g \in \text{Aut } G$, stabilizing φ). Then g is called H - φ -good if $\langle\langle c, g \rangle\rangle_\varphi = 1$ for all $c \in \mathbf{C}_{H/L}(g)$. We drop φ if it is clear from context. We also drop H if $H = G$.

By Lemma 2.1, $g \in G$ is $(H$ - φ -) good if and only if any other element of Lg is. Also if g is H -good, then any H -conjugate of g is H -good. We can thus speak of good conjugacy-classes of G/L . Lemma 2.2 has the following consequence:

2.5. Lemma. *If $h \in G$ and $\chi \in \mathbb{C}[\text{Irr}(G \mid \varphi)]$ are such that $\chi(h) \neq 0$, then h is good for φ .*

The following result is due to Gallagher [6]:

2.6. Lemma. *Let $\varphi \in \text{Irr } L$ be invariant in G , where $L \trianglelefteq G$. Then $|\text{Irr}(G \mid \varphi)|$ equals the number of φ -good conjugacy classes of G/L .*

For later use, we prove the following simple lemma which is essentially due to Isaacs [8, p. 600]:

2.7. Lemma. *Let $L \trianglelefteq G$ and $\varphi \in \text{Irr } L$ be invariant in G . Let $g \in G$ and $K \geq L$. If g^m is K -good, where $(m, |K/L|) = 1$, then g is K -good.*

Proof. For $c \in \mathbf{C}_{K/L}(g)$, we have $\langle\langle c, g \rangle\rangle^{|K/L|} = \langle\langle c^{|K/L|}, g \rangle\rangle = 1 = \langle\langle c, g^m \rangle\rangle = \langle\langle c, g \rangle\rangle^m$. Thus $\langle\langle c, g \rangle\rangle = 1$. \square

3. CHARACTER FIVES

First we remind the reader of some easy and well known equivalent conditions for a character to be fully ramified.

3.1. Lemma. *Let $L \trianglelefteq K$, $\varphi \in \text{Irr } L$ and $\vartheta \in \text{Irr}(K \mid \varphi)$. Then the following assertions are equivalent:*

- (i) $\vartheta_L = n\varphi$ with $n^2 = |K : L|$.
- (ii) $\varphi^K = n\vartheta$ with $n^2 = |K : L|$.
- (iii) φ is invariant in K and $\text{Irr}(K \mid \varphi) = \{\vartheta\}$.
- (iv) φ is invariant in K and ϑ vanishes outside L .
- (v) $e_\vartheta = e_\varphi$.
- (vi) φ is invariant in K and $\{L\}$ is the only φ -good conjugacy class of K/L .

Proof. By Lemma 2.6, $|\text{Irr}(K \mid \varphi)|$ equals the number of good conjugacy classes of K/L , when φ is invariant in K . This yields the equivalence of the third and the sixth condition. The equivalence of the other conditions is well known and easy to establish. \square

If φ has these properties, we say that φ is fully ramified in K . We remark that Howlett and Isaacs [7] have proved, using the classification of finite simple groups, that K/L is solvable if some $\varphi \in \text{Irr } L$ is fully ramified in K .

An interesting consequence of the last condition of the lemma is the following:

3.2. Corollary. *Suppose $\varphi \in \text{Irr } L$ is fully ramified in K , where K/L is abelian. Let e be the exponent of K/L . Then $\mathbb{Q}(\varphi)$ contains a primitive e -th root of unity.*

Proof. Since K/L is abelian, the form $\langle\langle \cdot, \cdot \rangle\rangle_\varphi$ is defined on all of $K/L \times K/L$. The last condition of the lemma implies that $\langle\langle \cdot, \cdot \rangle\rangle_\varphi$ is a nondegenerate alternating form on $K/L \times K/L$. Since it has values in $\mathbb{Q}(\varphi)^*$, this enforces $\mathbb{Q}(\varphi)$ to contain a primitive e -th root of unity. \square

Remark. Corollary 3.2 is false if K/L is not abelian: Namely, let C be a cyclic group of order p^{a+1} and let P be the Sylow p -subgroup of $\text{Aut } C$. Then $|P| = p^a$. Let K be the semidirect product of P and C . Then it is not difficult to see that $L = \mathbf{Z}(K) \subseteq P$ has order p and that the faithful characters of L are fully ramified in K . Clearly, K/L has exponent p^a .

In general, if p is a prime dividing $|K/L|$, then $\mathbb{Q}(\varphi)$ must contain the p -th roots of unity, and that is all that can be said.

3.3. Lemma. *Let $\varphi \in \text{Irr } L$ be fully ramified in K , where $L \trianglelefteq K$. Let \mathbb{F} be a field containing $\mathbb{Q}(\varphi)$. Then $S = (\mathbb{F}K e_\varphi)^L$ is central simple with dimension $|K/L|$ over \mathbb{F} .*

Proof. S is a twisted group algebra of K/L over \mathbb{F} (see the discussion before Lemma 2.3), that is, $S = \bigoplus_{x \in K/L} \mathbb{F} s_x$ and $\dim_{\mathbb{F}} S = |K/L|$. Since $\text{Irr}(K \mid \varphi)$ contains only one irreducible character, S must be simple. \square

The following definition describes the situation we will be concerned with in this paper:

3.4. Definition. A *character five* is a quintuple $(G, K, L, \vartheta, \varphi)$ where G is a finite group, $L \leq K$ are normal subgroups of G , and $\varphi \in \text{Irr } L$ is fully ramified in K , and $\{\vartheta\} = \text{Irr}(K \mid \varphi)$. Moreover, we assume that φ is invariant in G . An abelian (nilpotent, solvable) character five is a character five $(G, K, L, \vartheta, \varphi)$ with K/L abelian (nilpotent, solvable¹).

The term *character five* is due to Isaacs [8], but observe that he defines a character five to be abelian, and he only considers character fives where K/L is abelian. Since some of our results are valid when K/L is not abelian, we drop the hypothesis of commutativity of K/L from the definition of a character five. We hope that this change of terminology will not cause too much confusion.

4. CHARACTER CORRESPONDENCES FOR CHARACTER FIVES

Now let $(G, K, L, \vartheta, \varphi)$ be a character five and assume there exists a subgroup H such that $G = HK$ and $L = H \cap K$. Then $G/K \cong H/L$ canonically. Let \mathbb{F} be a field containing the values of φ (and thus of ϑ). We now review the theory of “magic representations” [15], that allows to construct an isomorphism $\varepsilon: \mathbf{M}_n(\mathbb{F}H e_\varphi) \rightarrow \mathbb{F}G e_\varphi$.

Let $S = (\mathbb{F}K e_\varphi)^L$. By Lemma 3.3, S is central simple. If \mathbb{F} is big enough (for example, if ϑ and φ are afforded by \mathbb{F} -representations), then $S \cong \mathbf{M}_n(\mathbb{F})$. Assume this and let $E = \{E_{ij} \mid i, j = 1, \dots, n\}$ be a full set of matrix units in S . (By this, we mean that $E_{ij}E_{kl} = \delta_{jk}E_{il}$ and $1_S = \sum_{i=1}^n E_{ii}$.) Set $A = \mathbb{F}G e_\varphi$. By a well known

¹Of course, by the before-mentioned result of Howlett and Isaacs [7], every character five is solvable.

ring theoretic result [16, pp. 17.4-17.6], we have that $A \cong \mathbf{M}_n(C)$, where $C = \mathbf{C}_A(E)$. It is clear that S as \mathbb{F} -space is generated by E , and thus $\mathbf{C}_A(E) = \mathbf{C}_A(S)$.

Write $A = \bigoplus_{x \in G/K} A_x$ with $A_{Kg} = \mathbb{F}Kge_\varphi$. This defines a grading of A . The subalgebra C inherits that grading: for $C_x = \mathbf{C}_{A_x}(S)$, we have $C = \bigoplus_{x \in G/K} C_x$. The above isomorphism is one of graded algebras:

4.1. Lemma. *When $S \cong \mathbf{M}_n(\mathbb{F})$, then $\mathbb{F}Ge_\varphi \cong \mathbf{M}_n(C)$ as G/K -graded algebras, where $C = \mathbf{C}_{\mathbb{F}Ge_\varphi}(S)$.*

The group G , and even G/L acts on $S = (\mathbb{F}Ke_\varphi)^L$ by conjugation. Let $x \in G/L$. Since S is central simple (by Lemma 3.3), the Skolem-Noether theorem yields that there is $\sigma(x) \in S^*$ with $s^x = s^{\sigma(x)}$ for all $s \in S$. Every such choice of $\sigma(x)$'s yields a projective representation $\sigma: G/L \rightarrow S$. It is unique up to multiplication with a map $G/L \rightarrow \mathbb{F}^*$. We sometimes speak of “the” projective representation associated with the character five $(G, K, L, \vartheta, \varphi)$. Let us recall the definition of a “magic representation” [15]:

4.2. Definition. Let $(G, K, L, \vartheta, \varphi)$ be a character five and $\mathbb{F} \supseteq \mathbb{Q}(\varphi)$. A *magic representation* is a map $\sigma: H/L \rightarrow S = (\mathbb{F}Ke_\varphi)^L$, where H/L is a complement of K/L in G/L , such that

- (a) $\sigma(x) \in S^*$,
- (b) $\sigma(xy) = \sigma(x)\sigma(y)$ for all $x, y \in H/L$ and
- (c) $s^x = s^{\sigma(x)}$ for all $s \in S$ and $x \in H/L$.

The character of a magic representation, that is the function $\psi: H/L \rightarrow \mathbb{F}$ with $\psi(x) = \text{tr}_{S/\mathbb{F}}(\sigma(x))$, is called a magic character.

Note that a magic representation is determined by the definition up to multiplication with a linear character of H/L .

4.3. Theorem. *Let $\sigma: H/L \rightarrow S$ be a magic representation. Then the linear map*

$$\kappa: \mathbb{F}H \rightarrow C = \mathbf{C}_{\mathbb{F}Ge_\varphi}(S), \quad \text{defined by } h \mapsto h\sigma(Lh)^{-1} \text{ for } h \in H,$$

is an algebra-homomorphism and induces an isomorphism $\mathbb{F}He_\varphi \cong C$. The isomorphism respects the H/L -grading of C and $\mathbb{F}He_\varphi$.

Proof. [15, Theorem 3.8] □

The reader should note that κ restricted to $\mathbb{F}L$ is just multiplication with e_φ , since $\sigma(1) = e_\varphi$. Using this, the proof of Theorem 4.3 is straightforward.

4.4. Corollary. *If there is a magic representation for the character five $(G, K, L, \vartheta, \varphi)$ and if $(\mathbb{F}Ke_\varphi)^L \cong \mathbf{M}_n(\mathbb{F})$, then $\mathbb{F}Ge_\varphi \cong \mathbf{M}_n(\mathbb{F}He_\varphi)$.*

4.5. Theorem. *Let $(G, K, L, \vartheta, \varphi)$ be a character five such that $S = (\mathbb{F}Ke_\varphi)^L \cong \mathbf{M}_n(\mathbb{F})$, where $\mathbb{Q}(\varphi) \leq \mathbb{F}$. Every magic representation $\sigma: H/L \rightarrow S^*$ determines linear isometries $\iota = \iota(\sigma)$ from $\mathbb{C}[\text{Irr}(U \mid \vartheta)]$ to $\mathbb{C}[\text{Irr}(U \cap H \mid \varphi)]$ for all U with $K \leq U \leq G$. If $E = \{E_{ij} \mid i, j = 1, \dots, n\}$ is a full set of matrix units in S , then $\iota(\sigma)$ can be computed by*

$$(1) \quad \chi^{\iota(\sigma)}(h) = \chi(E_{11}\sigma(Lh)^{-1}h).$$

Write ι also for the union of these isometries. Then ι commutes with restriction, induction and conjugation of class functions, with multiplication by class functions of $G/K \cong H/L$, and with field automorphisms fixing \mathbb{F} , and ι preserves Schur indices of irreducible characters over \mathbb{F} .

Let ψ be the character of σ , and $\chi \in \mathbb{C}[\text{Irr}(G \mid \vartheta)]$. Then

$$(2) \quad \chi_H = \psi\chi^\iota \quad \text{and} \quad (\chi^\iota)^G = \overline{\psi}\chi.$$

Proof. The theorem is a special case of Theorem 4.3 in [15]. \square

4.6. *Remark.* Let π be the set of prime divisors of $|K/L|$. If there is any magic representation, then there is a magic representation σ such that $\det \sigma$ has order a π -number.

Proof. [15, Remark 4.4] \square

5. MAGIC REPRESENTATIONS FOR CHARACTER FIVES

5.1. **Proposition.** *Let $L \trianglelefteq K$ and $\varphi \in \text{Irr } L$ be fully ramified in K . Let a be an element of some group acting on K such that $\varphi^a = \varphi$ and choose $\sigma \in S = (\mathbb{F}K e_\varphi)^L$ with $s^a = s^\sigma$ for all $s \in S$. Then*

$$\text{tr}(\sigma^{-1}) \text{tr}(\sigma) = \sum_{x \in \mathbf{C}_{K/L}(a)} \langle x, a \rangle_\varphi = \begin{cases} |\mathbf{C}_{K/L}(a)| & \text{if } a \text{ is } K\text{-good,} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The second equation is clear. Without loss of generality, we can assume that S splits, that is $S \cong \mathbf{M}_n(\mathbb{F})$. The \mathbb{F} -linear map κ from S to S sending s to $s^a = s^\sigma$ has trace $\text{tr}(\sigma^{-1}) \text{tr}(\sigma)$, as an easy computation with matrix units shows. Now we use as basis of S a set of graded units $s_x \in S \cap \mathbb{F}Lx$, $x \in K/L$. If $x \notin \mathbf{C}_{K/L}(a)$ then $(s_x)^a \in \mathbb{F}s_{x^a}$ is a multiple of another basis element and so it contributes nothing to the trace of κ . If $x \in \mathbf{C}_{K/L}(a)$, then $s_x^a = \langle x, a \rangle s_x$ by the remark following Lemma 2.3. The result now follows. \square

Applying the proposition to a magic representation of a character five yields the absolute value of a magic character. This generalizes a result of Isaacs [8, Theorem 3.5].

5.2. **Corollary.** *If $(G, K, L, \vartheta, \varphi)$ is a character five and ψ a magic character of this character five, defined on a complement H/L of K/L in G/L , then for $h \in H$*

$$|\psi(h)|^2 = \begin{cases} |\mathbf{C}_{K/L}(h)| & \text{if } h \text{ is } K\text{-good,} \\ 0 & \text{otherwise.} \end{cases}$$

Next we will show that there is, if the field is big enough, a finite group P , such that $S = (\mathbb{F}K e_\varphi)^L \cong \mathbb{F}P e_\mu$, where $\mu \in \text{Lin}(\mathbf{Z}(P))$. This follows of course at once from the theory of projective representations, but we need to take into account the action of G on S and so we review this in detail.

Remember that S has a natural grading $S = \bigoplus_{x \in K/L} S_x$ by the group K/L , and that each component has the form $S_x = \mathbb{F}s_x$, where s_x is a unit of S . In particular, $S_1 = \mathbb{F}e_\varphi = \mathbf{Z}(S)$. Let $\Omega = \bigcup_{x \in K/L} (S_x \cap S^*) = \bigcup_{x \in K/L} \mathbb{F}^* s_x$ be the set of graded units of S . Then Ω is a central extension of \mathbb{F}^* by K/L :

$$1 \longrightarrow \mathbb{F}^* \longrightarrow \Omega \xrightarrow{\varepsilon} K/L \longrightarrow 1.$$

Following Dade [2, 3], we call this central extension the Clifford extension associated with (K, L, φ) over \mathbb{F} . The epimorphism ε sends elements of S_x to $x \in K/L$. Note that for $u, v \in \Omega$ with $[u, v] \in \text{Ker } \varepsilon = \mathbb{F}^* e_\varphi$ we have $[u, v] = \langle u^\varepsilon, v^\varepsilon \rangle_\varphi$: this is Lemma 2.3. In particular, if φ is fully ramified in K , then $\text{Ker } \varepsilon = \mathbf{Z}(\Omega)$.

The group G (even G/L) acts on Ω and centralizes $\text{Ker } \varepsilon \cong \mathbb{F}^*$. Obviously, Ω generates S (as \mathbb{F} -algebra, even as ring), and so we might realize S as factor algebra of the group algebra $\mathbb{F}\Omega$. Of course, Ω is infinite.

5.3. **Lemma.** *Hold the above notation and let \mathbb{F} be algebraically closed. Set*

$$P = \langle s \in \Omega \mid \langle s \rangle \cap \mathbf{Z}(\Omega) = \{e_\varphi\} \rangle.$$

Then P has the following properties:

- (a) $P\mathbf{Z}(S)^* = \Omega$ (equivalently, the restriction of ε to P is surjective),
- (b) $P^g = P$ for all $g \in G$,
- (c) $|P|$ is finite and divides $|K/L|^2$.

Proof. Let $x \in K/L$ and $s \in S_x$. Then $s^{\mathbf{o}(x)} \in S_1 = \mathbb{F}e_\varphi$, so that $s^{\mathbf{o}(x)} = \lambda e_\varphi$. Since \mathbb{F} is algebraically closed, there is an $\mathbf{o}(x)$ -th root, α , of λ in \mathbb{F} . Thus $\alpha^{-1}s$ has indeed order $\mathbf{o}(x)$. This holds for any $x \in K/L$ and thus P covers K/L . It is also clear that P is invariant under G .

To see that P is finite, choose $s_x \in S_x$ with $\mathbf{o}(s_x) = \mathbf{o}(x)$ for any $x \in K/L$. Then P is generated by the s_x and the $\exp(K/L)$ -th roots of unity in \mathbb{F} . We have $s_x s_y = \alpha(x, y) s_{xy}$ for some $\alpha(x, y) \in \mathbb{F}$. Let $\delta(x)$ be the determinant of s_x (as element of $S \cong \mathbf{M}_n(\mathbb{F})$). Then $\delta(x)^n = 1$ since $\mathbf{o}(s_x) = \mathbf{o}(x)$ divides $n = \sqrt{|K/L|}$. Since $\delta(x)\delta(y) = \alpha(x, y)^n \delta(xy)$, it follows that $\alpha(x, y)^{n^2} = 1$. But $P \cap \mathbb{F}^*$ is generated by the values of α and thus is finite of order dividing $|K/L|$. This finishes the proof. \square

5.4. Definition. Let $(G, K, L, \vartheta, \varphi)$ be a character five and Ω the set of graded units of $S = (\mathbb{F}Ke_\varphi)^L$, where \mathbb{F} is some field containing the values of φ . Then an *admissible subgroup* for the character five $(G, K, L, \vartheta, \varphi)$ is a subgroup $P \leq \Omega$ having Properties (a)–(c) of Lemma 5.3.

If \mathbb{F} is not algebraically closed, then an admissible subgroup may or may not exist. There are, however, other conditions that ensure the existence of such a group (see Lemma 13.1 below).

5.5. Lemma. *Let P be an admissible subgroup of the character five $(G, K, L, \vartheta, \varphi)$ and set $Z = P \cap \mathbf{Z}(S)$. Let $\mu: Z \rightarrow \mathbb{F}^*$ be the restriction of ω_φ (the central character associated with φ) to Z . Then $P/Z \cong K/L$ canonically, $Z = \mathbf{Z}(P)$ and the inclusion $P \subset S$ induces an isomorphism $\mathbb{F}Pe_\mu \cong S$ of G -algebras.*

Proof. Z is the kernel of $P \rightarrow K/L$ and thus $P/Z \cong K/L$. That $Z = \mathbf{Z}(P)$ follows from $\mathbf{Z}(\Omega) = \mathbb{F}^*$ and $\Omega = P\mathbf{Z}(\Omega)$.

Note that $z = ze_\varphi = \mu(z)e_\varphi$ for $z \in Z$. The natural map $\mathbb{F}P \rightarrow S$ sends the central idempotent $e_\mu = (1/|Z|) \sum_{z \in Z} \mu(z^{-1})z$ of $\mathbb{F}P$ to e_φ , and sends all the other central idempotents of $\mathbb{F}Z$ to zero. As $\mathbb{F}P \rightarrow S$ is clearly surjective, S is isomorphic to a factor ring of $\mathbb{F}Pe_\mu$, but since $\dim_{\mathbb{F}} S = |K/L| = |P/Z| = \dim_{\mathbb{F}} \mathbb{F}Pe_\mu$, it follows that $\mathbb{F}Pe_\mu \cong S$. \square

Let us illustrate how this can be used.

5.6. Proposition. *Let $(G, K, L, \vartheta, \varphi)$ be a character five and suppose that $\sigma: H/L \rightarrow S = (\mathbb{C}Ke_\varphi)^L$ is a magic representation. Suppose that the order of $x \in H/L$ is relatively prime to $|K/L|$ and that $\det(\sigma(x)) = 1$. Then $\psi(x) = \text{tr}(\sigma(x))$ is rational.*

Proof. Let P be the group defined in Lemma 5.3 (over \mathbb{C}) and $\mu \in \text{Lin } \mathbf{Z}(P)$ be the linear character defined in Lemma 5.5.

Let $\mathbb{F} = \mathbb{Q}(\mu)$ and let T be the \mathbb{F} -subalgebra of $S = (\mathbb{C}Ke_\varphi)^L$ generated by P , so that $T \cong \mathbb{F}Pe_\mu$ naturally and $S = \mathbb{C}T \cong \mathbb{C} \otimes_{\mathbb{F}} T$. Then $T^x = T$. Since $\mathbf{o}(x)$ is prime to $\dim T$ and x acts on T , there is a unique element $\tau \in T$ such that the following conditions hold:

$$t^\tau = t^x \quad \text{for all } t \in T, \quad \tau^{\mathbf{o}(x)} = 1_T \quad \text{and} \quad \det(\tau) = 1.$$

The first condition is then in fact true for all $t \in S$, and τ is unique in S subject to these conditions. However, $\sigma(x)$ fulfills these conditions, so it follows that $\sigma(x) \in T$ and thus $\psi(x) \in \mathbb{F}$.

On the other hand, the eigenvalues of $\sigma(x)$ lie in a field \mathbb{E} obtained by adjoining a primitive $\mathbf{o}(x)$ -th root of unity to \mathbb{Q} , and thus $\psi(x) \in \mathbb{E}$. Since \mathbb{F} is obtained from \mathbb{Q}

by adjoining a $|Z(P)|$ -th root of unity, where $|Z(P)|$ divides $|K/L|$ (see Lemma 5.3), and since $(\mathbf{o}(x), |K/L|) = 1$, we get $\mathbb{F} \cap \mathbb{E} = \mathbb{Q}$ [17, Corollary on p. 204]. Thus $\psi(x) \in \mathbb{Q}$ as claimed. \square

6. DIGRESSION: COPRIME CHARACTER FIVES

6.1. Proposition. *Let $(G, K, L, \vartheta, \varphi)$ be a character five such that $|G : K|$ and $|K : L|$ are coprime. Then there is $H \leq G$ with $G = HK$ and $H \cap K = L$ and a unique magic character ψ of H/L of determinant 1. This character vanishes nowhere, so that the equation $\chi_H = \psi\xi$ defines an isometry between $\mathbb{C}[\text{Irr}(G \mid \vartheta)]$ and $\mathbb{C}[\text{Irr}(H \mid \varphi)]$. Moreover, ψ is rational.*

Proof. By the Schur-Zassenhaus Theorem, there is a complement H/L of K/L in G/L . Since $(|H/L|, n) = 1$, the action of H/L on $S = (\mathbb{Q}(\varphi)Ke_\varphi)^L$ lifts uniquely to a magic representation with determinant 1. Let ψ be its character. By Lemma 2.7, every $h \in H$ is K -good and thus $\psi(h) \neq 0$ for all $h \in H$ by Corollary 5.2. The character correspondence of Theorem 4.5 is determined by the equation $\chi_H = \psi\xi$ since ψ has no zeros. Finally, Proposition 5.6 yields that ψ is rational. \square

6.2. Remark. Suppose $x \in H/L$ has order p^r where p is a prime. Let $\omega \in \mathbb{C}$ be a primitive p^r -th root of 1. Then $\omega - 1 \in \mathfrak{P}$ for any prime ideal \mathfrak{P} of $\mathbb{Z}[\omega]$ with $\mathfrak{P} \cap \mathbb{Z} = p\mathbb{Z}$. It follows that $\psi(x) \equiv \psi(1) \pmod{\mathfrak{P}}$. This holds for any character and is well known. Since here $\psi(x)$ is rational, we even have that $\psi(x) \equiv \psi(1) \pmod{p}$. If p is an odd prime, then $\psi(x)$ is completely determined by the two conditions

$$\psi(x)^2 = |\mathbf{C}_{K/L}(x)| \quad \text{and} \quad \psi(x) \equiv n \pmod{p}.$$

We emphasize that we need only the character ψ to compute the correspondence: The correspondent of $\chi \in \text{Irr}(G \mid \vartheta)$ is $(1/\psi)\chi_H$ and the correspondent of $\xi \in \text{Irr}(H \mid \varphi)$ is $(1/\overline{\psi})\xi^G$. If $|K/L|$ is odd, even more can be said.

6.3. Corollary. *In the situation of Proposition 6.1 assume that $|K/L|$ is odd. Let H/L be a complement of K/L in G/L and ψ the unique magic character with $\det \psi = 1$. Then for every $U/L \leq H/L$ with $|U/L|$ odd, 1_U is the unique constituent of ψ_U with odd multiplicity.*

Proof. By Proposition 5.6 we know that the magic character ψ with $\det \psi = 1$ is rational. By Corollary 5.2, $|\psi(h)|^2 = |\mathbf{C}_{K/L}(h)|$ for all $h \in H$. Since $|K/L|$ is odd, $\psi(h) \in \mathbb{Z}$ is odd for all $h \in H$. For $U \leq H$ with $|U/L|$ odd, let $\beta = \psi_U - 1_U$, a generalized character of U with $L \leq \text{Ker } \beta$. For $\tau \in \text{Irr}(U/L)$ we have

$$|U/L|(\beta, \tau)_{U/L} = \sum_{u \in U/L} \beta(u)\overline{\tau(u)} \in 2\mathbb{Z}$$

since $\beta(u)$ is even for all $u \in U/L$. As $|U/L|$ is odd, we conclude that $(\beta, \tau)_{U/L}$ is even. Thus every $\tau \in \text{Irr}(U/L)$ occurs with even multiplicity in β . Thus 1_U occurs with odd multiplicity in $\psi = 1_U + \beta$, while all other constituents occur with even multiplicity, as claimed. \square

We remark that in the course of the proof we have shown that β can be divided by 2. For this we could have appealed to a more general result of Knörr [13, Proposition 1.1(iii)], but for the convenience of the reader we have repeated the simple argument here.

The following result includes two related results of Lewis [19, 20, Theorem A in both]:

6.4. Corollary. *In the situation of Proposition 6.1, assume that $|G : L|$ is odd, and let H/L be a complement of K/L in G/L . In the bijection of Proposition 6.1, $\chi \in \text{Irr}(G \mid \vartheta)$ and $\xi \in \text{Irr}(H \mid \varphi)$ correspond if and only if (χ_H, ξ) is odd.*

Proof. It follows from the last result that $\psi = 1 + 2\gamma$ for some character γ of H/L . From $\chi_H = \psi\xi$ we get $\chi_H = \xi + 2\gamma\xi$. Thus ξ is the only constituent of χ_H occurring with odd multiplicity. \square

In the next section we will see that we can remove the hypothesis of coprimeness when we add the hypothesis that K/L is abelian (and odd).

7. ODD ABELIAN CHARACTER FIVES

The main goal of this and the next section is to give alternative proofs of some results due to Isaacs [8].

7.1. Theorem. *Let $(G, K, L, \vartheta, \varphi)$ be an odd abelian character five.² Then there is $H \leq G$ with $G = HK$ and $L = H \cap K$, such that every element of H is K -good, and there is a magic representation $\sigma: H/L \rightarrow (\mathbb{C}Ke_\varphi)^L$.*

Remark. We will see later that there is even a magic representation $\sigma: H/L \rightarrow (\mathbb{Q}(\varphi)Ke_\varphi)^L$.

Proof of Theorem 7.1. We fix some notation needed in the proof. Set $S = (\mathbb{C}Ke_\varphi)^L$ and let Ω be the group of graded units of S (with respect to the K/L -grading of S). Let

$$P = \langle s \in \Omega \mid \langle s \rangle \cap \mathbf{Z}(\Omega) = 1 \rangle$$

be the subgroup of Ω defined in Lemma 5.3 and set $Z = \mathbf{Z}(P)$. Let $\mu: Z \rightarrow \mathbb{C}$ be the linear character with $z = \mu(z)e_\varphi$. Note that by Lemma 5.5, $S \cong \mathbb{C}Pe_\mu$. Let $A = \mathbf{C}_{\text{Aut } P}(Z)$. We split the proof into a series of lemmas.

7.2. Lemma. *There is $\tau \in A = \mathbf{C}_{\text{Aut } P}(Z)$ such that τ inverts P/Z and $\tau^2 = 1$, and such that for $I = \text{Inn } P$ and $U = \mathbf{C}_A(\tau)$ we have $A = IU$ and $I \cap U = 1$.*

Proof. Note that every coset of Z in P contains by definition of P an element, p , with $\mathbf{o}(p) = \mathbf{o}(Zp)$. Now a result of Isaacs [8, Corollary 4.3] applies. (The proof is neither long nor difficult.) \square

Note that the action of G on P centralizes $\mathbf{Z}(S)$ and thus Z , and so we have a homomorphism $\kappa: G \rightarrow A$. Clearly, L is in the kernel of κ . The following observation is true for admissible subgroups of arbitrary character fives:

7.3. Lemma. *We have $K\kappa = \text{Inn } P$ and $K \cap \text{Ker } \kappa = L$.*

Proof. Let $k \in K$. Then there is $p \in P \cap \mathbb{C}Lk$. As $\mathbb{C}L$ centralizes S , we have $s^k = s^p$ for all $s \in S$, in particular for $s \in P$. Thus $k\kappa \in \text{Inn } P$. Conversely, every $p \in P$ is contained in $\mathbb{C}Lk$ for some $k \in K$, so that the inner automorphism of P induced by p comes from conjugation with $k \in K$. This shows $K\kappa = \text{Inn } P$. Therefore, $K/(K \cap \text{Ker } \kappa) \cong \text{Inn } P \cong P/Z$. It follows that $K \cap \text{Ker } \kappa = L$. \square

We keep the notation $I = \text{Inn } P$ and $U = \mathbf{C}_A(\tau)$, with τ as in Lemma 7.2. As before, $\kappa: G \rightarrow A$ is the homomorphism induced by the action of G on P .

7.4. Lemma. *Set $H = \kappa^{-1}(U)$. Then $G = HK$ and $H \cap K = L$.*

Proof. Since $(H \cap K)\kappa \subseteq U \cap I = 1$, it follows that $H \cap K \subseteq \text{Ker } \kappa \cap K = L$. Thus $H \cap K = L$. Since $I = K\kappa \leq G\kappa \leq A = UI$, it follows that $G\kappa = (U \cap G\kappa)I$ and thus $G = HK$. \square

7.5. Lemma. *All elements of H are K - φ -good.*

²This means that K/L is abelian of odd order

Proof. Let $h \in H$. We have to show that $\langle\langle x, h \rangle\rangle_\varphi = 1$ for all $x \in \mathbf{C}_{K/L}(h)$. First we translate this to a statement about the action of h on P . Let $s_x \in S_x$. Then by Lemma 2.3, $\langle\langle x, h \rangle\rangle_\varphi = s_x^{-1} s_x^h$. We may choose $s_x \in P$, since $P \rightarrow K/L$ is surjective. Also observe that the isomorphism $P/Z \cong K/L$ sends $\mathbf{C}_{P/Z}(h)$ onto $\mathbf{C}_{K/L}(h)$. Thus we need to show that $[c, h] = 1$ for all $c \in C$, where $C \leq P$ is defined by $C/Z = \mathbf{C}_{P/Z}(h)$. We may replace h by $u = h\kappa \in U$. Since $u^\tau = u$, it follows $C^\tau = C$. Since $\mathbf{C}_{P/Z}(\tau) = 1$, we have $C = [C, \tau]$ as sets. Thus every element in C has the form $[d, \tau] = d^{-1}d^\tau$ for some $d \in C$. For these,

$$\begin{aligned} [d^{-1}d^\tau, u] &= [d^{-1}, u][d^\tau, u] = [d, u]^{-1}[d^\tau, u^\tau] \\ &= [d, u]^{-1}[d, u] = 1, \end{aligned}$$

as was to be shown. \square

7.6. Lemma. *There is a representation $W: U \rightarrow \mathbb{C}Pe_\mu$ such that $s^u = s^{W(u)}$ for all $s \in \mathbb{C}Pe_\mu$.*

This is, in principle, well known. Namely, the group P can be interpreted as a Heisenberg group, and W is the corresponding Weil representation [cf. 1, 24]. We give a proof for the sake of completeness and to show that the result is neither very deep nor difficult.

Proof of Lemma 7.6. There is a projective representation $\widetilde{W}: U \rightarrow T = \mathbb{C}Pe_\mu$ such that $s^u = s^{\widetilde{W}(u)}$ for all $s \in T$. Set $t = \widetilde{W}(\tau)$, where τ is as in Lemma 7.2. As $\mathfrak{o}(\tau) = 2$, we may assume, after replacing t by a suitable scalar multiple, that $t^2 = 1_T$. Let $u \in U$. Since $\tau^u = \tau$, it follows that $t^{\widetilde{W}(u)}$ is a scalar multiple of t . On the other hand, we have $(\text{tr } t)^2 = |\mathbf{C}_{P/Z}(\tau)| = 1$ by Proposition 5.1. In particular, $\text{tr}(t) \neq 0$. It follows $t^{\widetilde{W}(u)} = t$.

Now let V be a simple T -module and set

$$V_+ = \{v \in V \mid vt = v\} \quad \text{and} \quad V_- = \{v \in V \mid vt = -v\}.$$

Then, since $t^2 = 1$, we have $V = V_+ \oplus V_-$. From the previous paragraph it follows that V_+ and V_- are invariant under $\widetilde{W}(u)$. Set $d_+ = \dim V_+$ and $\delta_+(u) = \det(\widetilde{W}(u) \text{ on } V_+)$, and define d_- and $\delta_-(u)$ similarly. From

$$\widetilde{W}(u)\widetilde{W}(v) = \alpha(u, v)\widetilde{W}(uv)$$

we get

$$\delta_\pm(u)\delta_\pm(v) = \alpha(u, v)^{d_\pm} \delta_\pm(uv).$$

Since $(\text{tr } t)^2 = 1$, it follows $\text{tr } t = \pm 1 = d_+ - d_-$. We may assume $d_+ - d_- = -1$. Set

$$W(u) = \frac{\delta_+(u)}{\delta_-(u)} \widetilde{W}(u).$$

It is now easy to verify that W is multiplicative. The lemma follows. \square

The proof of Theorem 7.1 is finished by noting that

$$H/L \xrightarrow{\kappa} U \xrightarrow{W} \mathbb{C}Pe_\mu \longrightarrow S$$

is the desired magic representation. \square

8. THE CANONICAL MAGIC CHARACTER

In the odd abelian case, it is possible to choose a canonical ψ , as Isaacs has shown. The existence and the most important properties of this canonical magic character can be derived from what we have done so far, with (I hope) simpler proofs. Some of the arguments we need are taken from the original proof, but for the convenience of the reader and the sake of completeness we repeat them here. The following is an adaption of Isaacs' definition of "canonical" [8, Definition 5.2] to our purposes.

8.1. Definition. Let $(G, K, L, \vartheta, \varphi)$ be a character five with $|K/L|$ odd. Let π be the set of prime divisors of $|K/L|$. A magic character $\psi \in \mathbb{Z}[\text{Irr}(H/L)]$ is called *canonical* if

- (a) $\mathbf{o}(\psi)$ is a π -number and
- (b) If $p \in \pi$ and $Q \in \text{Syl}_p(H)$, then 1_Q is the unique irreducible constituent of ψ_Q which appears with odd multiplicity.

8.2. Remark. Let $(G, K, L, \vartheta, \varphi)$ be a character five with $|K/L|$ odd. If a canonical magic character $\psi: H \rightarrow \mathbb{C}$ exists, then all $h \in H$ are good.

Proof. Let $h \in H$. We have to show that $\langle\langle h, c \rangle\rangle_\varphi = 1$ for all $c \in \mathbf{C}_{K/L}(h)$. Write $h = \prod_p h_p$ as product of its p -parts. Since $\mathbf{C}_{K/L}(h) = \bigcap_p \mathbf{C}_{K/L}(h_p)$, we may assume that h itself has prime power order, p^t , say. If p does not divide $|K/L|$, then h is good by Lemma 2.7. If p divides $|K/L|$, then let $Q \in \text{Syl}_p(H)$ be a Sylow p -subgroup containing h . Then, by canonicalness, $\psi_Q = 1_Q + 2\beta$ for some character β . It follows that $\psi(h) \neq 0$ and thus h is good by Corollary 5.2. \square

If K/L is not abelian, it may happen that there is no canonical ψ even if there is a magic character. For example it may be that there are p -elements in a complement that are not good. An example where this occurs has been given by Lewis [18]. In his example, K/L is a p -group, and the complement H is unique up to conjugacy.

8.3. Lemma. *The complement H given, there is at most one canonical magic character ψ .*

Proof. [8, p. 610] Suppose ψ and ψ_1 are canonical. Then $\psi_1 = \lambda\psi$ for some $\lambda \in \text{Lin}(H/L)$. If some prime $q \notin \pi$ divides $\mathbf{o}(\lambda)$, then from $\det \psi_1 = \lambda^n \det \psi$ and $(q, n) = 1$ we conclude that q divides $\mathbf{o}(\psi_1)$, but this contradicts ψ_1 being canonical. Therefore $\mathbf{o}(\lambda)$ is a π -number. Let $p \in \pi$ and $Q \in \text{Syl}_p H$. Then $[\lambda_Q, (\psi_1)_Q] = [1_Q, \psi_Q]$ and the last is odd by the definition of canonical. From the assumption that ψ_1 is canonical too we conclude that $\lambda_Q = 1_Q$. This holds in fact for all π -subgroups of H . As $\mathbf{o}(\lambda)$ is a π -number, we have $\lambda = 1_H$ and $\psi_1 = \psi$. \square

8.4. Theorem. *If $(G, K, L, \vartheta, \varphi)$ is an odd abelian character five, then there is a canonical magic character ψ . Let H/L be the complement of K/L in G/L on which ψ is defined. For every subgroup $V/L \leq H/L$ with $|V/L|$ odd, 1_V is the unique irreducible constituent of ψ which appears with odd multiplicity.*

Proof. (cf. Isaacs [8, Theorem 5.3].) Let $P, A, \tau \in A$ and $U = \mathbf{C}_A(\tau)$ be as in the proof of Theorem 7.1. It suffices to show that we may choose the $W: U \rightarrow \mathbb{C}P e_\mu$ in Lemma 7.6 such that its character, which we still call ψ , is canonical.

First, let W be any representation as in Lemma 7.6, and let ψ be its character. We can assume that ψ has π -order. As $\tau \in \mathbf{Z}(U)$, we can write $\psi = \psi_+ + \psi_-$ where $\psi_+(\tau) = \psi_+(1)$ and $\psi_-(\tau) = -\psi_-(1)$.

Let $V \leq U$ with $|V|$ odd and take $v \in V$. As τ centralizes v and v has odd order, we have $\tau = (\tau v)^{\mathbf{o}(v)}$. Thus $\mathbf{C}_{P/Z}(\tau v) \subseteq \mathbf{C}_{P/Z}(\tau) = 1$. Therefore

$$1 = |\psi(\tau v)| = |\psi_+(v) - \psi_-(v)|$$

for every $v \in V$. This yields $(\psi_+ - \psi_-, \psi_+ - \psi_-)_V = 1$ and hence $(\psi_+ - \psi_-)_V = \pm \lambda$ where $\lambda \in \text{Lin } V$. The sign depends not on V , but only on whether $\psi_+(1) > \psi_-(1)$ or $\psi_+(1) < \psi_-(1)$. We conclude

$$\psi_V = 2\gamma_V + \lambda, \text{ where } \gamma = \begin{cases} \psi_- & \text{if } \psi_+(1) > \psi_-(1) \\ \psi_+ & \text{if } \psi_+(1) < \psi_-(1). \end{cases}$$

This equation shows that λ is the only constituent of ψ_V occurring with odd multiplicity. Taking determinants in the equation yields $\det \psi_V = (\det \gamma_V)^2 \lambda$. Thus λ can be extended to a linear character of U , namely to $\mu = \det \psi (\det \gamma)^{-2}$. Write $\mu = \mu_\pi \mu_{\pi'}$ where μ_π is the π -part of μ . Then $\overline{\mu_\pi} \psi$ still has determinantal order a π -number. For $Q \in \text{Syl}_p(U)$ where $p \in \pi$ we have $(\mu_\pi)_Q = \mu_Q$ and thus the unique irreducible constituent of $\overline{\mu_\pi} \psi$ with odd multiplicity is 1_Q . This shows that $\overline{\mu_\pi} \psi$ is canonical and completes the proof of the existence of a canonical magic character.

Now assume that ψ is canonical. We have already seen that for $|V|$ odd, ψ has a unique constituent λ of odd multiplicity and that this constituent is linear. To show that $\lambda = 1_V$ it suffices to show that $\lambda_Q = 1_Q$ if Q is a p -subgroup of V . If $p \in \pi$ this is clear from the definition of canonical. If $p \notin \pi$, then $(|Q|, |P/Z|) = 1$ and the result follows from Corollary 6.3, applied to Q . \square

As in the coprime case, we get as a corollary:

8.5. Corollary. *Let $(G, K, L, \vartheta, \varphi)$ be an abelian character five with $|G : L|$ odd. Then there is a complement H/L of K/L in G/L and a bijection between $\text{Irr}(G \mid \vartheta)$ and $\text{Irr}(H \mid \varphi)$ where χ and ξ correspond if and only if (χ_H, ξ) is odd.*

We conclude this section with some results showing that the canonical magic representation has values in $(\mathbb{Q}(\varphi)Ke_\varphi)^L$.

If α is an automorphism of the field \mathbb{E} , then α acts on the group algebra $\mathbb{E}G$ by acting on coefficients. If $\sigma : H/L \rightarrow (\mathbb{E}Ke_\varphi)^L$ is a magic representation, then we write σ^α for the map $H/L \ni x \mapsto \sigma(x)^\alpha \in (\mathbb{E}Ke_{\varphi^\alpha})^L$. Now the following is easy to verify [15, Proposition 4.5]:

8.6. Proposition. *Let $(G, K, L, \vartheta, \varphi)$ be a character five with magic representation σ and magic character ψ , and let α be a field automorphism. Then σ^α is a magic representation with magic character ψ^α for the character five $(G, K, L, \vartheta^\alpha, \varphi^\alpha)$.*

The definition of a canonical character is invariant under field automorphisms. Thus:

8.7. Corollary. *Let $(G, K, L, \vartheta, \varphi)$ be an odd abelian character five with canonical magic character ψ , and let α be a field automorphism. Then ψ^α is the canonical magic character associated with the character five $(G, K, L, \vartheta^\alpha, \varphi^\alpha)$.*

8.8. Corollary. *The image of the canonical magic representation is contained in $(\mathbb{Q}(\varphi)Ke_\varphi)^L$, and the values of the canonical character are in $\mathbb{Q}(\varphi)$.*

Proof. Let $\sigma : H/L \rightarrow (\mathbb{C}Ke_\varphi)^L$ be the canonical magic representation. Let α be a field automorphism fixing $\mathbb{Q}(\varphi)$. Thus $\varphi^\alpha = \varphi$. By Proposition 8.6, σ^α is a magic representation for the character five $(G, K, L, \vartheta, \varphi)$. Since σ^α is canonical too, we have $\sigma^\alpha = \sigma$. Since this holds for all α centralizing $\mathbb{Q}(\varphi)$, it follows that $\sigma(Lh) \in \mathbb{Q}(\varphi)K$, as was to be shown. The second assertion follows from the first. \square

Remark. In fact, one can show that the values of the canonical character are contained in a much smaller field [8, Cor. 6.4, or 14, Cor. 4.35].

It follows that Theorem 4.5 applies with $\mathbb{F} = \mathbb{Q}(\varphi)$, if we know that $(\mathbb{Q}(\varphi)Ke_\varphi)^L \cong \mathbf{M}_n(\mathbb{Q}(\varphi))$.

9. SEMI-INVARIANT CHARACTERS

We review the notion of semi-invariant characters and recall some results that we need. Throughout this section, let $L \trianglelefteq G$ and $\varphi \in \text{Irr } L$. Let \mathbb{F} be a field of characteristic zero. All characters are assumed to take values in some field containing \mathbb{F} , so that expressions like $\mathbb{F}(\varphi)$ are defined. We need the following well known fact.

9.1. **Lemma.**

$$e = T_{\mathbb{F}}^{\mathbb{F}(\varphi)}(e_{\varphi}) := \sum_{\alpha \in \text{Gal}(\mathbb{F}(\varphi)/\mathbb{F})} (e_{\varphi})^{\alpha}$$

is the unique central primitive idempotent of $\mathbb{F}L$ for which $\varphi(\mathbb{F}Le) \neq 0$.

9.2. Notation. We write $e_{(\varphi, \mathbb{F})}$ for the idempotent of Lemma 9.1. In particular, if $\mathbb{F} = \mathbb{F}(\varphi)$, then $e_{(\varphi, \mathbb{F})} = e_{\varphi}$.

9.3. **Lemma.**

$$\mathbb{F}G_{\varphi}e_{(\varphi, \mathbb{F})} \ni a \mapsto ae_{\varphi} \in \mathbb{F}(\varphi)G_{\varphi}e_{\varphi}$$

is an isomorphism of \mathbb{F} -algebras.

Proof. [15, Lemma 5.3] □

The following notation will be convenient: Let $L \trianglelefteq G$ and $e \in \mathbf{Z}(\mathbb{F}L)$ be a primitive idempotent. Let $G_e = \{g \in G \mid e^g = e\}$ and write e^G for the idempotent defined by $e^G := T_{G_e}^G(e) = \sum_{g \in [G:G_e]} e^g$. Finally, given an idempotent e of $\mathbf{Z}(\mathbb{F}L)$, we set

$$\text{Irr}(G \mid e) = \{\chi \in \text{Irr } G \mid \chi(e) \neq 0\}.$$

The following result is also well known:

9.4. Proposition. Set $e = e_{(\varphi, \mathbb{F})}$ and $f = e^G$ and let $T = G_e$ be the inertia group of e . Then $\mathbb{F}Gf \cong \mathbf{M}_{|G:T|}(\mathbb{F}Te)$. Induction defines a bijection between $\text{Irr}(T \mid e)$ and $\text{Irr}(G \mid f)$ that preserves field of values and Schur indices over \mathbb{F} .

In general, G_{φ} may be smaller than $G_e = T$. For $\xi \in \text{Irr}(G_{\varphi} \mid \varphi)$, the field $\mathbb{F}(\xi^T)$ is contained in $\mathbb{F}(\xi)$, but may be strictly smaller. If this happens, the Schur index of ξ^T over $\mathbb{F}(\xi^T)$ may be bigger than that of ξ .

9.5. Definition. Let $L \trianglelefteq G$ and $\varphi \in \text{Irr } L$. We say that φ is *semi-invariant* in G over the field \mathbb{F} , if $e_{(\varphi, \mathbb{F})}$ is invariant in G . If φ is semi-invariant over \mathbb{Q} , then we say it is semi-invariant.

This definition is equivalent to the one given by Isaacs [9, Definition 1.1] (cf. [15, Lemma 5.6]).

9.6. Lemma. Let $L \trianglelefteq G$ and $\varphi \in \text{Irr } L$ be semi-invariant over \mathbb{F} . Set $\Gamma = \text{Gal}(\mathbb{F}(\varphi)/\mathbb{F})$.

- (a) For every $g \in G$ there is a unique $\alpha_g \in \Gamma$ such that $\varphi^{g\alpha_g} = \varphi$.
- (b) The map $g \mapsto \alpha_g$ is a group homomorphism from G into Γ with kernel G_{φ} .
- (c) For $g \in G$ and $z \in \mathbf{Z}(\mathbb{F}L)$ we have $\omega_{\varphi}(z^g) = \omega_{\varphi}(z)^{\alpha_g}$, where $\omega_{\varphi}: \mathbf{Z}(\mathbb{F}L) \rightarrow \mathbb{F}(\varphi)$ is the central character associated with φ .

Proof. [9, Lemma 2.1], [15, Lemma 5.7] □

10. MAIN THEOREM

For convenience, we introduce some terminology.

10.1. Definition. A quintuple $(G, K, L, \vartheta, \varphi)$ is called a semi-invariant character five, if G is a finite group, $L \leq K$ are normal subgroups of G , and the characters $\varphi \in \text{Irr } L$ and $\vartheta \in \text{Irr } K$ are fully ramified with respect to each other and semi-invariant in G .

As the attentive reader will have remarked, this terminology is inconsistent with Definition 3.4, since a semi-invariant character five is not necessarily a character five in the sense of Definition 3.4. It would have been more consistent to speak of “invariant character fives”, “semi-invariant character fives” and “character fives” (not necessarily semi-invariant). To avoid any ambiguity, we will speak of invariant/semi-invariant character fives from now on.

We will need one further hypothesis.

10.2. Definition. A semi-invariant character five $(G, K, L, \vartheta, \varphi)$ with K/L abelian is said to be *strongly controlled* if there is $N \leq G$, such that the following hold:

- (a) $K \leq N \leq G_\varphi$,
- (b) $\mathbf{C}_{K/L}(N) = 1$,
- (c) $(|N/\mathbf{C}_N(K/L)|, |K/L|) = 1$.

Thus the subgroup of the automorphism group of K/L induced by N acts coprimely and fixed point freely on K/L . I do not know whether such an assumption is really necessary for the next result. On the other hand, the assumption that $|K/L|$ is odd is necessary, even for strongly controlled character fives.

10.3. Theorem. Let $(G, K, L, \vartheta, \varphi)$ be a strongly controlled semi-invariant character five, such that K/L is abelian of odd order. Set $f = e_{(\varphi, \mathbb{Q})}$. Then there is $H \leq G$ such that $KH = G$, $K \cap H = L$, every element of H_φ is K - φ -good, and $\mathbb{Q}Gf \cong \mathbf{M}_n(\mathbb{Q}Hf)$ as G/K -graded algebras (with $n = \sqrt{|K/L|} = \vartheta(1)/\varphi(1)$).

Suppose $K \leq U \leq G$ and set $V = U \cap H$. The isomorphism of the theorem restricts to an isomorphism $\mathbb{Q}Uf \cong \mathbf{M}_n(\mathbb{Q}Vf)$. These isomorphisms yield character correspondences. (See the discussion in [15, Section 2] or [22, Theorem 3.4].) While there is no canonical choice for the isomorphism of Theorem 10.3, there are choices that lead to a canonical choice of the bijection between $\mathbb{C}[\text{Irr}(G | f)]$ and $\mathbb{C}[\text{Irr}(H | f)]$. (The isomorphism of Theorem 10.3 is unique up to (inner) isomorphisms of $(\mathbb{Q}Kf)^L \cong \mathbf{M}_n(\mathbb{Q}(\varphi))$.)

10.4. Proposition. Assume the situation of Theorem 10.3. For every $U \leq G$ with $K \leq U$, there is an isomorphism

$$\iota: \mathbb{C}[\text{Irr}(U | f)] \rightarrow \mathbb{C}[U \cap H | f].$$

The union ι of these isomorphisms has the following properties:

- (a) $\chi \in \text{Irr}(U | f)$ if and only if $\chi^\iota \in \text{Irr}(U \cap H | f)$.
- (b) $(\chi_1^\iota, \chi_2^\iota)_{U \cap H} = (\chi_1, \chi_2)_U$ for $\chi_1, \chi_2 \in \mathbb{C}[\text{Irr}(U | f)]$.
- (c) $\chi(1) = n\chi^\iota(1)$.
- (d) $(\chi_{U_1})^\iota = (\chi^\iota)_{U_1 \cap H}$ for $K \leq U_1 \leq U_2 \leq G$ and $\chi \in \mathbb{C}[\text{Irr}(U_2 | f)]$.
- (e) $(\tau^{U_2})^\iota = (\tau^\iota)^{U_2 \cap H}$ for $K \leq U_1 \leq U_2 \leq G$ and $\tau \in \mathbb{C}[\text{Irr}(U_1 | f)]$.
- (f) $(\chi^\iota)^h = (\chi^h)^\iota$ for $h \in H$.
- (g) $(\beta\chi)^\iota = \beta\chi^\iota$ for all $\beta \in \mathbb{C}[\text{Irr}(U/K)]$.
- (h) If α is a field automorphism, then $(\chi^\alpha)^\iota = (\chi^\iota)^\alpha$; and $\mathbb{Q}(\chi) = \mathbb{Q}(\chi^\iota)$.
- (i) $[\chi]_{\mathbb{Q}} = [\chi^\iota]_{\mathbb{Q}}$ for $\chi \in \text{Irr}(U | f)$.
- (j) If $U \leq G_\varphi$ and $\chi \in \text{Irr}(U | \varphi)$, then $\chi_{U \cap H} = \psi\chi^\iota$, where ψ is the canonical magic character associated with the invariant character five $(G_\varphi, K, L, \vartheta, \varphi)$.

Moreover, ι is determined uniquely by some of these properties (namely, by linearity and (e), (h) and (j)).

Proof (Uniqueness). Let ι and ι_1 be two such isometries and take a subgroup U with $K \leq U \leq G$ and $\chi \in \text{Irr}(U \mid \varphi)$. By Clifford correspondence, $\chi = \tau^U$ for a unique $\tau \in \text{Irr}(U_\varphi \mid \varphi)$. By (e), it follows that $\chi^\iota = (\tau^U)^\iota = (\tau^\iota)^U$ and similarly for ι_1 . By (j), we have $\tau^\iota = (1/\psi)\tau_{U_\varphi \cap H} = \tau^{\iota_1}$. Thus $\chi^\iota = \chi^{\iota_1}$. Finally, for $\alpha \in \text{Aut}(\mathbb{Q}(\varphi))$ any element of $\text{Irr}(U \mid \varphi^\alpha)$ has the form χ^α for some $\chi \in \text{Irr}(U \mid \varphi)$, and we have $(\chi^\alpha)^\iota = (\chi^\iota)^\alpha = (\chi^{\iota_1})^\alpha = (\chi^\alpha)^{\iota_1}$. Since $\text{Irr}(U \mid f) = \bigcup_{\alpha \in \text{Aut}(\mathbb{Q}(\varphi))} \text{Irr}(U \mid \varphi^\alpha)$, it follows that $\iota = \iota_1$. This shows uniqueness. \square

The existence will be proved in Section 13, together with Theorem 10.3. In Section 14, we use these results to show that the Isaacs half of the Glauberman-Isaacs correspondence preserves Schur indices (among other things).

11. EXISTENCE AND UNIQUENESS OF THE COMPLEMENT

In this section, we show that if a semi-invariant character five is strongly controlled, then there is a complement H/L to K/L in G/L , and that H is determined up to conjugacy by some weak additional condition. This result will not be needed in the proof of Theorem 10.3 and Proposition 10.4. In fact, in the proof of the latter results, we will give another construction of the supplement H .

We need a general lemma about the bilinear form associated with φ .

11.1. Lemma. *Let $L \trianglelefteq G$, let $\varphi \in \text{Irr } G$ and $x, y \in G_\varphi$ with $[x, y] \in L$.*

- (a) *If $g \in G$, then $\langle\langle x^g, y^g \rangle\rangle_{\varphi^g} = \langle\langle x, y \rangle\rangle_\varphi$.*
- (b) *If $\alpha \in \text{Aut } \mathbb{Q}(\varphi)$, then $\langle\langle x, y \rangle\rangle_{\varphi^\alpha} = \langle\langle x, y \rangle\rangle_\varphi^\alpha$.*

Proof. Let $\mathbb{E} = \mathbb{Q}(\varphi)$. Remember that $\langle\langle x, y \rangle\rangle_{\varphi e_\varphi} = [x, y][c_y, c_x]$, where $c_x \in \mathbb{E}Le_\varphi$ is such that $a^x = a^{c_x}$ for all $a \in \mathbb{E}Le_\varphi$, and similar for c_y . If $g \in G$, then $c_x^g \in (\mathbb{E}Le_\varphi)^g = \mathbb{E}Le_{\varphi^g}$. Any $b \in \mathbb{E}Le_{\varphi^g}$ can be written as $b = a^g$ with $a \in \mathbb{E}Le_\varphi$. Thus

$$b^{c_x^g} = (a^g)^{c_x^g} = a^{c_x g} = a^{xg} = a^{g x^g} = b^{x^g}.$$

It follows that

$$\begin{aligned} \langle\langle x^g, y^g \rangle\rangle_{\varphi^g e_{\varphi^g}} &= [x^g, y^g][c_y^g, c_x^g] = ([x, y][c_y, c_x])^g \\ &= (\langle\langle x, y \rangle\rangle_{\varphi e_\varphi})^g = \langle\langle x, y \rangle\rangle_{\varphi e_{\varphi^g}}. \end{aligned}$$

The first assertion follows. The proof of the second is similar: We may extend $\alpha_\mathbb{E}$ naturally to an automorphism of $\mathbb{E}G$, acting trivially on G . Then we get

$$\begin{aligned} \langle\langle x, y \rangle\rangle_{\varphi^\alpha e_{\varphi^\alpha}} &= [x, y][c_y^\alpha, c_x^\alpha] = ([x, y][c_y, c_x])^\alpha \\ &= (\langle\langle x, y \rangle\rangle_{\varphi e_\varphi})^\alpha = \langle\langle x, y \rangle\rangle_\varphi^\alpha e_{\varphi^\alpha}. \end{aligned}$$

The proof follows. \square

The arguments in the proof of the next result extend those of Isaacs [10, p. 304–5] for invariant character fives.

11.2. Proposition. *Let $(G, K, L, \vartheta, \varphi)$ be a strongly controlled character five with K/L abelian. Then there is a unique conjugacy class of subgroups $H \leq G$ such that $HK = G$, $H \cap K = L$ and every element of $H \cap \mathbf{C}_N(K/L)$ is K - φ -good.*

Proof. Let $C = \mathbf{C}_N(K/L)$. Observe that then $\langle\langle \cdot, \cdot \rangle\rangle_\varphi$ is defined on $C/L \times K/L$. Let

$$B = \{c \in C \mid \langle\langle c, k \rangle\rangle_\varphi = 1 \text{ for all } k \in K\}.$$

We claim that $B \trianglelefteq G$. Let $b \in B$ and $g \in G$. There is $\alpha \in \text{Aut}(\mathbb{Q}(\varphi))$ such that $\varphi^{\alpha g} = \varphi$. Let $k \in K$. Using both parts of Lemma 11.1, we get

$$\langle\langle b^g, k^g \rangle\rangle_\varphi = \langle\langle b^g, k^g \rangle\rangle_{\varphi^{\alpha g}} = \langle\langle b, k \rangle\rangle_{\varphi^\alpha} = \langle\langle b, k \rangle\rangle_\varphi^\alpha = 1.$$

Since $k^g \in K^g = K$ was arbitrary, it follows that $b^g \in B$. This establishes that $B \trianglelefteq G$.

Via $\langle\langle \cdot, \cdot \rangle\rangle_\varphi$, the factor group C/B is isomorphic to a subgroup of $\text{Lin}(K/L)$, and so $|C/B| \leq |K/L|$. Since φ is fully ramified in K/L , the form $\langle\langle \cdot, \cdot \rangle\rangle_\varphi$ is nondegenerate on K/L , and thus we have $B \cap K = L$. Therefore $|K/L| = |BK/B| \leq |C/B|$. It follows $BK = C$ and $C/B \cong K/L$. Since $|C/B| = |K/L|$ and $|N/C|$ are coprime, the group C/B has a complement, M/B , in N/C . Let $H = \mathbf{N}_G(M)$. By the Frattini-argument, $G = HN = HMC = HBK = HK$. Moreover, we have $[H \cap K, M] \leq K \cap M = L$, and so $(H \cap K)/L \leq \mathbf{C}_{K/L}(M) = \mathbf{C}_{K/L}(N/C) = 1$, so that $H \cap K = L$. Since $K/L \cong C/B$ as group with M -action, the same argument shows that $H \cap C = B$. By definition of B , every element of B is K - φ -good.

For uniqueness, assume that U is another subgroup having the properties in the proposition. Then $U \cap C \leq B$, since $U \cap C$ is good. Since $C = C \cap UK = (C \cap U)K$, it follows that $C \cap U = B$. Since $N = N \cap UC = (N \cap U)C$, it follows that $(N \cap U)/B$ is a complement of C/B in N/B . By the conjugacy part of the Schur-Zassenhaus Theorem, $N \cap U = M^c$ with $c \in C$. It follows that $H^c = \mathbf{N}_G(M^c) \geq U$, and thus $\mathbf{N}_G(M^c) = U$ as claimed. \square

In the special case where $(|N/K|, |K/L|) = 1$, the supplement H is determined up to conjugacy by the properties $HK = G$ and $H \cap K = L$. This can be proved using standard, purely group theoretical arguments and is well known.

12. MAGIC CROSSED REPRESENTATIONS FOR SEMI-INVARIANT CHARACTER FIVES

We need to review the theory of magic crossed representations, which we will use to prove the results of Section 10. For convenient reference, let us fix the following assumptions and notation:

12.1. Hypothesis.

- (a) $(G, K, L, \vartheta, \varphi)$ is a semi-invariant character five,
- (b) $f = e_{(\varphi, \mathbb{Q})}$, $\mathbb{E} = \mathbf{Z}(\mathbb{Q}Kf)$ and $S = (\mathbb{Q}Kf)^L$,
- (c) for $g \in G$, let $\alpha_g \in \text{Aut } \mathbb{E}$ be the automorphism of \mathbb{E} induced by conjugation with g , and set $\mathbb{F} = \mathbb{E}^G$.

The isomorphism $\mathbb{Q}Kf \cong \mathbb{Q}(\varphi)Ke_\varphi$ of Lemma 9.3, given by $a \mapsto ae_\varphi$, sends $\mathbf{Z}(\mathbb{Q}Kf)$ onto $\mathbf{Z}(\mathbb{Q}(\varphi)Ke_\varphi) = \mathbb{Q}(\varphi)e_\varphi$. The same is true for the centers of $\mathbb{Q}Lf$ and S . Thus $\mathbf{Z}(\mathbb{Q}Kf) = \mathbf{Z}(\mathbb{Q}Lf) = \mathbf{Z}(S) \cong \mathbb{Q}(\varphi)$. The combination of Lemma 9.3 and Lemma 3.3 yields that S is central simple over \mathbb{E} .

G acts on \mathbb{E} by conjugation. By Lemma 9.6, Part (c), the kernel of this action is G_φ , the inertia group of φ . Thus we have proved:

12.2. Lemma. *We have $\mathbb{E} = \mathbf{Z}(\mathbb{Q}Lf) = \mathbf{Z}(S) \cong \mathbb{Q}(\varphi)$, and S is central simple over \mathbb{E} . G acts on \mathbb{E} with kernel G_φ .* \square

To prove Theorem 10.3, we will procede as follows: First, we show that $S \cong \mathbf{M}_n(\mathbb{E})$. Thus we find a full set of matrix units, E , in S . Then $\mathbb{Q}Gf \cong \mathbf{M}_n(C)$, where $C = \mathbf{C}_{\mathbb{Q}Gf}(E)$ is the centralizer of E in $\mathbb{Q}Gf$. Second, we show that $C \cong \mathbb{Q}Hf$, where H/L is a complement of K/L in G/L . To do this, we need a magic crossed representation, which generalizes a magic representation to the semi-invariant case [15]. We review this concept now.

Clearly, $\text{Aut } \mathbb{E}$ acts naturally on $\mathbf{M}_n(\mathbb{E})$ by acting on the entries of a matrix. Thus, if an isomorphism $S \cong \mathbf{M}_n(\mathbb{E})$ is given, it yields an action of $\text{Aut } \mathbb{E}$ on S . To be more concrete, let $\{e_{ij} \mid 1 \leq i, j \leq n\}$ be a set of matrix units in S . For $\alpha \in \text{Aut } \mathbb{E}$, define $\hat{\alpha} \in \text{Aut } S$ by $\left(\sum_{i,j} s_{ij} e_{ij}\right)^{\hat{\alpha}} = \sum_{i,j} s_{ij}^{\alpha} e_{ij}$ for $s_{ij} \in \mathbb{E}$. For convenient reference, we summarize our assumptions and notation:

12.3. Hypothesis. Assume Hypothesis 12.1 and the following:

- (a) $H \leq G$ with $G = HK$ and $H \cap K = L$,
- (b) $S \cong \mathbf{M}_n(\mathbb{E})$,
- (c) $\text{Aut } \mathbb{E} \ni \alpha \mapsto \hat{\alpha} \in \text{Aut } S$ denotes the action of $\text{Aut } \mathbb{E}$ on S with respect to a fixed set of matrix units E in S .

Now let us recall the definition of a magic crossed representation, adapted to our situation:

12.4. Definition. In the situation of 12.3, $\sigma: H/L \rightarrow S$ is a *magic* crossed representation (with respect to $h \mapsto \hat{\alpha}_h$), if, for all $u, v \in H/L$ and $s \in S$, we have

$$\sigma(u)^{\hat{\alpha}_v} \sigma(v) = \sigma(uv) \quad \text{and} \quad s^u = s^{\hat{\alpha}_v \sigma(u)}.$$

If $\sigma: H/L \rightarrow S$ is a magic crossed representation, then the linear map

$$\kappa: \mathbb{Q}H \rightarrow C = \mathbf{C}_{\mathbb{Q}Gf}(E), \quad \text{defined by} \quad h \mapsto h\sigma(Lh)^{-1} \text{ for } h \in H,$$

is an algebra-homomorphism and induces an isomorphism $\mathbb{Q}Hf \cong C$ [15, Theorem 6.11]. Thus, since $\mathbb{Q}Gf \cong \mathbf{M}_n(C)$, Theorem 10.3 will follow if we can show that there is a magic crossed representation.

Moreover, every magic crossed representation determines a character correspondence $\iota(\sigma)$ between $\text{Irr}(U \mid f)$ and $\text{Irr}(U \cap H \mid f)$ for each $K \leq U \leq G$ having Properties (a) to (i) of Proposition 10.4 [15, Theorem 6.13]. Finally, if $\sigma: H/L \rightarrow S$ is a magic crossed representation, then the map $\sigma_{\varphi}: H_{\varphi} \rightarrow (\mathbb{Q}(\varphi)Ke_{\varphi})^L$ defined by $\sigma_{\varphi}(Lh) = \sigma(Lh)e_{\varphi}$ is a magic representation for the invariant character five $(G_{\varphi}, K, L, \vartheta, \varphi)$, and for $\chi \in \text{Irr}(U_{\varphi} \mid \varphi)$ we have $(\chi^U)^{\iota(\sigma)} = (\chi^{\iota(\sigma_{\varphi})})^{U \cap H}$ [15, Proposition 6.15]. Here $\iota(\sigma_{\varphi})$ means the character correspondence of Theorem 4.5. Thus to prove Theorem 10.3 and Proposition 10.4, it suffices to show the following:

12.5. Theorem. *Assume the situation of Theorem 10.3. Then the assumptions of Hypothesis 12.3 hold, and there is a magic crossed representation $\sigma: H/L \rightarrow S$ such that $\sigma_{\varphi}: H_{\varphi}/L \rightarrow (\mathbb{Q}(\varphi)Ke_{\varphi})^L$, defined by $\sigma_{\varphi}(h) = \sigma(h)e_{\varphi}$, is the canonical magic representation of the invariant character five $(G_{\varphi}, K, L, \vartheta, \varphi)$.*

13. PROOF OF MAIN THEOREM

The proof of Theorem 12.5 mimics that of Theorem 7.1. Assume Hypothesis 12.1. Then $S = (\mathbb{Q}Kf)^L$ admits a natural grading by the factor group K/L :

$$S = \bigoplus_{x \in K/L} S_x \quad \text{with} \quad S_{Lk} = S \cap \mathbb{Q}Lk.$$

Moreover, every component S_x contains units of S . All this follows from the corresponding results for invariant character fives, via Lemma 9.3. Let

$$\Omega = \bigcup_{x \in K/L} (S_x \cap S^*)$$

be the set of graded units of S . Then we have a central extension

$$1 \longrightarrow \mathbb{E}^* \longrightarrow \Omega \xrightarrow{\varepsilon} K/L \longrightarrow 1.$$

The group G acts on Ω . Since φ is only semi-invariant, the action of G on $\mathbf{Z}(\Omega) = \mathbb{E}^*$ may be nontrivial.

Now assume the situation of Theorem 12.5. Remember that we are given a semi-invariant character five, $(G, K, L, \vartheta, \varphi)$, with K/L abelian of odd order, which is strongly controlled, that is, there is $K \leq N \trianglelefteq G$ such that $N/\mathbf{C}_N(K/L)$ acts coprimely and fixed point freely on K/L . First we exhibit a subgroup $P \leq \Omega$ with similar properties as in Lemma 5.3. This is the only part of the proof of Theorem 12.5 where we use the assumption that the character five is strongly controlled. The main idea in the proof of the following lemma is taken from a paper of Turull [28].

13.1. Lemma. *Set*

$$P = [\Omega, N] \quad \text{and} \quad Z = P \cap \mathbb{E}.$$

Then the following hold:

- (a) $P\mathbf{Z}(S)^* = \Omega$ (equivalently, $P \rightarrow K/L$ is surjective).
- (b) $P^g = P$ for all $g \in G$.
- (c) $|P|$ is finite and has the same exponent as K/L .
- (d) Every coset of Z in P contains an element u with $Z \cap \langle u \rangle = \{1\}$.

Proof. Since $N \trianglelefteq G$, it follows that G normalizes P , this is (b).

The group N acts on Ω and centralizes $\mathbb{E} = \mathbf{Z}(S)$, since $N \leq G_\varphi$. Since $N/\mathbf{C}_N(K/L)$ acts coprimely and fixed point freely on $K/L \cong \Omega/\mathbb{E}^*$, we have $[\Omega/\mathbb{E}^*, N] = \Omega/\mathbb{E}^*$. It follows that $[\Omega, N]\mathbb{E}^* = \Omega$, as claimed.

Let k be an odd natural number. The set $\{x \in K/L \mid x^k = 1\}$ is a characteristic subgroup of K/L and thus normalized by N . Let

$$\Omega_k = \{u \in \Omega \mid u^k \in \mathbb{E}\}$$

be the pre-image in Ω . Since N acts coprimely and fixed point freely on Ω_k/\mathbb{E}^* , it follows that $[\Omega_k, N]\mathbb{E}^* = \Omega_k$. Thus

$$P \cap \Omega_k = P \cap [\Omega_k, N]\mathbb{E}^* = [\Omega_k, N](P \cap \mathbb{E}^*) = [\Omega_k, N]Z.$$

Next we claim that $u^k = 1$ for every $u \in [\Omega_k, N]$.

First, let $s, t \in \Omega_k$ be arbitrary. Remember that $[\cdot, \cdot]: \Omega \times \Omega \rightarrow \mathbb{E}^*$ is bilinear. Thus

$$(st)^k = s^k t^k [s, t]^{\binom{k}{2}} = s^k t^k [s^{\binom{k}{2}}, t] = s^k t^k$$

as k is odd.

Let $a \in N$ and $s \in \Omega_k$. We apply the last equation to s^{-1} and $t = s^a$:

$$[s, a]^k = (s^{-1} s^a)^k = s^{-k} (s^a)^k = s^{-k} (s^k)^a = 1,$$

as $a \in N$ centralizes $s^k \in \mathbb{E}^*$. It follows that $[\Omega_k, N]$ is generated by elements of order k . Since we saw $(st)^k = s^k t^k$ before, it follows that $u^k = 1$ for all $u \in [\Omega_k, N]$, as claimed.

If we take for k the exponent of K/L , then $\Omega_k = \Omega$ and thus $P = [\Omega, N]$ has the same exponent as K/L . It follows that Z is finite (and cyclic). Thus $|P| = |K/L||Z|$ is finite, too.

Now let $v \in P$ be arbitrary and set $k = \mathbf{o}(Zv)$. We have seen earlier that $P \cap \Omega_k = [\Omega_k, N]Z$. Thus there is $u \in Zv \cap [\Omega_k, N]$, and we have seen before that u has order k . This means that $Z \cap \langle u \rangle = 1$, which shows (d). \square

13.2. Lemma. *Let $P \leq \Omega$ be a group satisfying Properties (a)–(c) of Lemma 13.1, and set $Z = P \cap \mathbb{E}$. Let $\mu: Z \rightarrow \mathbb{Q}(\varphi)$ be the restriction of the central character of φ to Z . Then*

- (a) *The set $\{ue_\varphi \mid u \in P\} \subseteq (\mathbb{Q}(\varphi)Ke_\varphi)^L$ is an admissible subgroup for the character five $(G_\varphi, K, L, \vartheta, \varphi)$.*

- (b) $Z = \mathbf{Z}(P)$ and $P/Z \cong K/L$ as groups with G -action.
- (c) $S \cong \mathbb{Q}(\varphi)Pe_\mu$.
- (d) The commutator map defines a non-degenerate alternating form $P/Z \times P/Z \rightarrow Z$.

Proof. Part (a) follows from applying the isomorphism of Lemma 9.3. Part (c) and $Z = \mathbf{Z}(P)$ then follows from Lemma 5.5. That the isomorphism $P/Z \cong K/L$ respects the action of G is clear. Finally, the form $\langle\langle \cdot, \cdot \rangle\rangle: K/L \times K/L \rightarrow \mathbb{Q}(\varphi)$ is non-degenerate. Let $x, y \in K/L$ be the images of $u, v \in P$ under the canonical homomorphism $\varepsilon: P \rightarrow K/L$. Then, by Lemma 2.3,

$$\langle\langle x, y \rangle\rangle_\varphi e_\varphi = [ue_\varphi, ve_\varphi] = [u, v]e_\varphi = \mu([u, v])e_\varphi.$$

It follows that $(u, v) \mapsto \mu([u, v])$ is non-degenerate, and thus $[\cdot, \cdot]$ itself is non-degenerate. \square

In the next result, $\text{Aut } S$ denotes the set of the ring automorphisms of S , which are the automorphisms of S as \mathbb{Q} -algebra. We could also work with the automorphisms of S as \mathbb{F} -algebra (where $\mathbb{F} = \mathbf{C}_\mathbb{E}(G)$).

13.3. Lemma. *Let P be as in Lemma 13.1. The action of G on S defines an homomorphism*

$$\kappa: G \rightarrow A := \{\alpha \in \text{Aut } S \mid P^\alpha = P\}$$

with L in the kernel. Let

$$I = \{\alpha \in \mathbf{C}_A(\mathbb{E}) \mid \alpha|_P \in \text{Inn } P\}.$$

Then $K\kappa = I \cong \text{Inn } P$.

Proof. The first assertion is clear.

The map $\mathbf{C}_A(\mathbb{E}) \rightarrow \text{Aut } P$, $\alpha \mapsto \alpha|_P$, is injective, since \mathbb{E} and P generate S as ring and $A \leq \text{Aut } S$. By definition, I maps into $\text{Inn } P$. Conversely, every inner automorphism of P induces an inner automorphism of S , simply since $P \leq S^*$, and thus centralizes $\mathbb{E} = \mathbf{Z}(S)$. Thus $I \cong \text{Inn } P$. If $k \in K$, then for every unit $u \in S_{Lk}$ we have $s^k = s^u$ for all $s \in S$, since L centralizes S . It follows $K\kappa = I$. \square

13.4. Lemma. *Let A and I be as in Lemma 13.3. Then I is the kernel of the natural map $A \rightarrow \text{Aut}(P/Z) \times \text{Aut } \mathbb{E}$.*

Proof. Since $P/Z \cong K/L$ is abelian, inner automorphisms of P centralize P/Z . Thus I is in the kernel of $A \rightarrow \text{Aut}(P/Z) \times \text{Aut}(\mathbb{E})$. Conversely, suppose $\alpha \in A$ acts trivially on P/Z and on \mathbb{E} . Then α centralizes also $Z \subset \mathbb{E}$. It is known [8, Lemma 4.2] and not difficult to show that an automorphism of P centralizing P/Z and Z is inner. (Here one needs that P is a $*$ -group in the sense of Isaacs [8, Def 4.1], which follows from Lemma 13.1, Part (d).) Thus $\alpha \in I$ as claimed. \square

The next lemma generalizes Lemma 7.2 to our situation, the proof is nearly the same.

13.5. Lemma. *There is $\tau \in A = \mathbf{N}_{\text{Aut } S}(P)$ such that τ inverts P/Z , centralizes \mathbb{E} , $\tau^2 = 1$, and $U = \mathbf{C}_A(\tau)$ is a complement of I in A .*

Proof. There is $\tau_0 \in \text{Aut } P$ of order 2, inverting P/Z and centralizing Z (Lemma 7.2 [cf. 8, Lemma 4.3]). Since $S \cong \mathbb{Q}(\varphi)Pe_\mu$, this τ_0 can be extended to an automorphism τ of S of order 2 and centralizing \mathbb{E} .

Observe that τ maps to a central element of $\text{Aut}(P/Z) \times \text{Aut } \mathbb{E}$. Thus $I\tau \in \mathbf{Z}(A/I)$ and so $\langle I, \tau \rangle \leq A$. Since $I \cong P/Z$ has odd order, $\langle \tau \rangle \in \text{Syl}_2(I, \tau)$, and thus, by the Frattini argument, $A = I\mathbf{C}_A(\tau)$. As τ inverts P/Z , it follows $\mathbf{C}_I(\tau) = 1$, as desired. \square

13.6. Corollary. *Let $H = \kappa^{-1}(U)$. Then $G = HK$ and $H \cap K = L$. Every element of H_φ is K -good.*

Proof. From $G\kappa \leq A = UI$ and $I = K\kappa \leq G\kappa$ it follows $G\kappa = (G\kappa \cap U)I$ and thus $G = HK$. If $k \in H \cap K$, then $k\kappa \in U \cap I = 1$, and thus $s^k = s$ for all $s \in S$. It follows $k \in L$.

That elements of H_φ are good follows from the corresponding result for invariant φ (Lemma 7.5). \square

Note that by Proposition 11.2, the complement H is unique up to conjugacy. We will prove Theorem 12.5 for the group H of the last corollary. We now work toward finding a suitable set of matrix units in S .

13.7. Lemma. *Let $R = \{r \in P \mid r^\tau = r^{-1}\}$ with τ the automorphism of Lemma 13.5. Then $P = \bigcup_{r \in R} Zr$.*

Proof. For arbitrary $x \in P$ one has $x^\tau = zx^{-1}$ for some $z \in Z$. There is a unique $d \in Z$ with $d^2 = z$ as Z has odd order. For this d we get $d^{-1}x \in R$. Thus every coset of Z contains exactly one element of R , as claimed. \square

13.8. Lemma. *There are abelian subgroups $X, Y \leq P$ with $X \cap Y = Z$ and $XY = P$. Set*

$$e = \frac{1}{n} \sum_{x \in X \cap R} x \in S.$$

Then $e^\tau = e$ and

$$\{E_{r,s} = r^{-1}es \mid r, s \in R \cap Y\}$$

is a full set of matrix units in S . (Thus $S \cong \mathbf{M}_n(\mathbb{E})$.)

Proof. Remember that $[\ , \] : P/Z \times P/Z \rightarrow Z$ is a nondegenerate alternating form (Lemma 13.2(d)). Choose two maximal isotropic subspaces, X/Z and Y/Z , with $P/Z = X/Z \times Y/Z$. Then X and Y are abelian and we have $X \cap Y = Z$ and $XY = P$.

As X is abelian, it follows that $R \cap X$ is a subgroup: We have $(rs)^\tau = r^{-1}s^{-1} = (rs)^{-1}$ for $r, s \in R \cap X$. The order of $X \cap R$ is $|X/Z| = n$. It follows that e is an idempotent. That $e^\tau = e$ is clear.

Next, let $y \in Y \setminus Z$. We claim that $e^y e = 0$. Note that the group algebra $\mathbb{E}[X \cap R]$ is contained canonically in S as subalgebra, since $S = \bigoplus_{r \in R} \mathbb{E}r$. We may view e and

$$e^y = \frac{1}{n} \sum_{x \in X \cap R} x^y = \frac{1}{n} \sum_{x \in X \cap R} x[x, y]$$

as idempotents in $\mathbb{E}[X \cap R]$, since $[x, y] \in \mathbb{E}$. If $y \in Y \setminus Z$, then $x \mapsto [x, y] \in \mathbb{E}^*$ is a nontrivial group homomorphism from $X \cap R$ to \mathbb{E}^* . It follows that $e^y e = 0$ and $e y e = 0$. Thus we get

$$E_{r,s} E_{u,v} = r^{-1} s e^s e^u u^{-1} v = \delta_{s,u} r^{-1} e v = \delta_{s,u} E_{r,v}$$

for $r, s, u, v \in Y \cap R$. We also get

$$\sum_{r \in Y \cap R} E_{r,r} = \sum_{r \in Y \cap R} \frac{1}{n} \sum_{x \in X \cap R} x[x, r] = \frac{1}{n} \sum_{x \in X \cap R} x \sum_{r \in Y \cap R} [x, r] = 1_S.$$

The result follows. \square

Note that the isomorphism $S \cong \mathbf{M}_n(\mathbb{E})$ can be used to define an action of $\text{Aut } \mathbb{E}$ on S . The point about the next lemma is that the corresponding action homomorphism has image in $U \leq \text{Aut } S$, where $U = \mathbf{C}_A(\tau)$, as defined in Lemma 13.5.

13.9. Lemma. For $\alpha \in \text{Aut } \mathbb{E}$, define $\hat{\alpha} \in \text{Aut } S$ by

$$\left(\sum_{r,s \in Y \cap R} c_{r,s} E_{r,s} \right)^{\hat{\alpha}} = \sum_{r,s \in Y \cap R} c_{r,s}^{\alpha} E_{r,s} \quad \text{for } c_{r,s} \in \mathbb{E}.$$

Then $\alpha \mapsto \hat{\alpha}$ is a monomorphism from $\text{Aut } \mathbb{E}$ into $U = \mathbf{C}_A(\tau)$, and $\hat{\alpha}_{\mathbb{E}} = \alpha$.

Proof. It is clear that $\alpha \mapsto \hat{\alpha}$ is a monomorphism into $\text{Aut } S$. From

$$\left(\sum_{r,s \in Y \cap R} c_{r,s} E_{r,s} \right)^{\tau} = \sum_{r,s \in Y \cap R} c_{r,s} E_{r^{-1},s^{-1}}$$

we see that $\hat{\alpha}\tau = \tau\hat{\alpha}$. Thus $\hat{\alpha} \in \mathbf{C}_{\text{Aut } S}(\tau)$. It remains to show that $\hat{\alpha} \in A$, that is, $\hat{\alpha}$ maps P onto itself. We do this by showing that $\hat{\alpha}$ maps Z , $Y \cap R$ and $X \cap R$ onto itself.

It is clear that $\hat{\alpha}$ maps Z onto itself, since $\hat{\alpha}_{\mathbb{E}} = \alpha$ and Z is a finite subgroup of \mathbb{E}^* .

Let y and $r \in Y \cap R$. Then $ye^r = yr^{-1}er = E_{ry^{-1},r}$. Thus

$$y^{\hat{\alpha}} = \left(\sum_{r \in Y \cap R} ye^r \right)^{\hat{\alpha}} = \left(\sum_{r \in Y \cap R} E_{ry^{-1},r} \right)^{\hat{\alpha}} = \sum_{r \in Y \cap R} E_{ry^{-1},r} = y,$$

so in fact $\hat{\alpha}$ centralizes $Y \cap R$.

Now let $x \in X \cap R$ and $r \in Y \cap R$. Then

$$\begin{aligned} xe^r &= \frac{1}{n} \sum_{u \in X \cap R} xu[u, r] = \frac{1}{n} \sum_{u \in X \cap R} u[x^{-1}u, r] \\ &= [x^{-1}, r] \frac{1}{n} \sum_{u \in X \cap R} u[u, r] = [r, x]e^r, \end{aligned}$$

with $[r, x] \in Z$. (Remember that the commutator map is bilinear in both variables.) As Z is a finite subgroup of \mathbb{E} , there is $k \in \mathbb{N}$ with $z^{\alpha} = z^k$ for all $z \in Z$. Thus

$$\begin{aligned} x^{\hat{\alpha}} &= \left(\sum_{r \in R \cap Y} xe^r \right)^{\hat{\alpha}} = \sum_{r \in R \cap Y} ([r, x]e^r)^{\hat{\alpha}} = \sum_{r \in R \cap Y} [r, x]^k e^r \\ &= \sum_{r \in R \cap Y} [r, x^k]e^r = x^k. \end{aligned}$$

Thus $\hat{\alpha}$ maps $X \cap R$ onto itself. This finishes the proof that $P^{\hat{\alpha}} = P$. \square

Let $U_{\varphi} = \mathbf{C}_U(\mathbb{E})$ with $U = \mathbf{C}_A(\tau)$ as in Lemma 13.5, and observe that $H_{\varphi} = \kappa^{-1}(U_{\varphi})$. For $u \in U$, we denote by α_u the restriction of u to $\mathbb{E} = \mathbf{Z}(S)$. To prove Theorem 12.5, we will show the following:

13.10. Lemma. There is a map $\sigma: U \rightarrow S$ such that for all $u, v \in U$

$$\sigma(u)^{\hat{\alpha}_v} \sigma(v) = \sigma(uv) \quad \text{and} \quad s^u = s^{\hat{\alpha}_u \sigma(u)} \quad \text{for all } s \in S,$$

and such that $\sigma_{U_{\varphi}}: U_{\varphi} \rightarrow S$ is canonical in the sense of Definition 8.1.

Then Theorem 12.5 follows by composing σ and $\kappa: H \rightarrow U$.

Proof of Lemma 13.10. From the results of Section 7 it follows that there is an homomorphism $\sigma_{\varphi}: U_{\varphi} \rightarrow (\mathbb{C}Ke_{\varphi})^L$ that is magic in the sense that $s^u = s^{\sigma_{\varphi}(u)}$ for all $s \in (\mathbb{C}Ke_{\varphi})^L$ and $u \in U_{\varphi}$ (Lemma 7.6), and we may assume that σ_{φ} is canonical in the sense of Definition 8.1 by the results of Section 8. By Corollary 8.8, the image of σ is contained in $(\mathbb{Q}(\varphi)Ke_{\varphi})^L$. Now remember that $S \ni s \mapsto se_{\varphi} \in (\mathbb{Q}(\varphi)Ke_{\varphi})^L$

is an isomorphism (Lemma 9.3). We get a unique homomorphism $\sigma_{U_\varphi} : U_\varphi \rightarrow S$ such that $\sigma_{U_\varphi}(u)e_\varphi = \sigma_\varphi(u)$ for all $u \in U_\varphi$. Moreover, for $s \in S$ and $u \in U_\varphi$ we have $s^u = s^{\sigma_{U_\varphi}(u)}$, again by Lemma 9.3. We must extend σ_{U_φ} to a magic crossed representation of U .

For $u \in U$ and $z \in \mathbb{E}$, we have $z^{\widehat{\alpha}_u} = z^u$ and thus $\widehat{\alpha}_u^{-1}u \in U_\varphi$. Now define

$$\sigma(u) := \sigma_{U_\varphi}(\widehat{\alpha}_u^{-1}u).$$

Since $U_\varphi = \text{Ker}(u \mapsto \alpha_u)$, the map σ extends σ_{U_φ} . For $s \in S$,

$$s^{\widehat{\alpha}_u \sigma(u)} = s^{\widehat{\alpha}_u \sigma_{U_\varphi}(\widehat{\alpha}_u^{-1}u)} = s^{\widehat{\alpha}_u(\widehat{\alpha}_u^{-1}u)} = s^u.$$

To see that σ is a crossed representation, we need the following fact:

$$(*) \quad \sigma_{U_\varphi}(u)^a = \sigma_{U_\varphi}(u^a) \quad \text{for arbitrary } u \in U_\varphi \text{ and } a \in U.$$

Assuming this for the moment, we see that

$$\begin{aligned} \sigma(u)^{\widehat{\alpha}_v} \sigma(v) &= \sigma_{U_\varphi}(\widehat{\alpha}_u^{-1}u)^{\widehat{\alpha}_v} \sigma_{U_\varphi}(\widehat{\alpha}_v^{-1}v) \\ &= \sigma_{U_\varphi}\left(\left(\widehat{\alpha}_u^{-1}u\right)^{\widehat{\alpha}_v}\right) \sigma_{U_\varphi}(\widehat{\alpha}_v^{-1}v) \\ &= \sigma_{U_\varphi}(\widehat{\alpha}_v^{-1}\widehat{\alpha}_u^{-1}uv) = \sigma_{U_\varphi}(\widehat{\alpha}_{uv}^{-1}uv) = \sigma(uv), \end{aligned}$$

where the second equation follows from (*), applied to $a = \widehat{\alpha}_v \in U$.

To establish (*), view $a \in U$ as fixed and consider the map $u \mapsto (\sigma_{U_\varphi})^a(u) = \sigma_{U_\varphi}(u^{a^{-1}})^a$. We will show that $(\sigma_{U_\varphi})^a$ is also a canonical magic representation. From uniqueness it will then follow that $(\sigma_{U_\varphi})^a = \sigma_{U_\varphi}$, that is, $\sigma_{U_\varphi}(u)^a = (\sigma_{U_\varphi})^a(u^a) = \sigma_{U_\varphi}(u^a)$ as claimed.

Clearly $(\sigma_{U_\varphi})^a$ is a homomorphism.

Let $s^a \in S$. Then

$$(s^a)^{(\sigma_{U_\varphi})^a(u)} = (s^a)^{\sigma_{U_\varphi}(aua^{-1})^a} = (s^{\sigma_{U_\varphi}(aua^{-1})})^a = (s^{aua^{-1}})^a = (s^a)^u.$$

Thus $(\sigma_{U_\varphi})^a$ is magic.

Let ψ be the character of σ_{U_φ} with values in \mathbb{E} on which U acts. Then ψ^a defined by $\psi^a(u^a) = \psi(u)^a$ is the character of $(\sigma_{U_\varphi})^a$. The definition of “canonicalness” (Definition 8.1) is invariant to conjugation by group automorphisms and field automorphisms, and thus ψ^a is canonical. It follows that $(\sigma_{U_\varphi})^a$ is magic and canonical and thus $(\sigma_{U_\varphi})^a = \sigma_{U_\varphi}$. The claim, (*), follows. \square

This finishes the proofs of Theorem 10.3 and Proposition 10.4.

14. ISAACS CORRESPONDENCE AND SCHUR INDICES

14.1. Theorem. *Let $L, K \trianglelefteq G$ with $L \leq K$ and K/L of odd order. Suppose there is $M \leq G$ with $MK \trianglelefteq G$, $(|M/L|, |K/L|) = 1$ and $\mathbf{C}_{K/L}(M) = 1$. Let $\varphi \in \text{Irr}_M L$ and $\vartheta \in \text{Irr}_M K$ with $(\vartheta_L, \varphi) > 0$ and set $H = \mathbf{N}_G(M)$. Set $e = (e_{(\vartheta, \mathbb{Q})})^G$ and $f = (e_{(\varphi, \mathbb{Q})})^H$. Then $\mathbb{Q}Ge \cong \mathbf{M}_n(\mathbb{Q}Hf)$, where $n = \vartheta(1)/\varphi(1)$. There is a canonical bijection between $\text{Irr}(G | e)$ and $\text{Irr}(H | f)$ which commutes with field automorphisms and preserves Schur indices. (More precisely, the correspondence has Properties (a)–(i) of Proposition 10.4.)*

We need the following version of the going-down theorem [12, Theorem 6.18] for semi-invariant characters:

14.2. Proposition. *Let K/L be an abelian chief factor of G and suppose $\vartheta \in \text{Irr } K$ is \mathbb{F} -semi-invariant in G for some field $\mathbb{F} \subseteq \mathbb{C}$. Then one of the following holds:*

- (a) $\vartheta = \varphi^K$ with $\varphi \in \text{Irr } L$, and either
 - (i) $\mathbb{F}(\varphi) = \mathbb{F}(\vartheta)$, or
 - (ii) $K/L \cong \text{Gal}(\mathbb{F}(\varphi)/\mathbb{F}(\vartheta))$ and φ is \mathbb{F} -semi-invariant in G .
- (b) $\vartheta_L = \varphi \in \text{Irr } L$.
- (c) $\vartheta_L = e\varphi$ with $\varphi \in \text{Irr } L$ and $e^2 = |K/L|$, and $\mathbb{F}(\vartheta) = \mathbb{F}(\varphi)$.

Proof. Let φ be an irreducible constituent of ϑ_L . Let

$$T = \{g \in G \mid \varphi^g \text{ is Galois conjugate to } \varphi \text{ over } \mathbb{F}\}.$$

Let $g \in G$ and pick $\alpha \in \text{Gal}(\mathbb{F}(\vartheta)/\mathbb{F})$ with $\vartheta^g = \vartheta^\alpha$. Then

$$((\vartheta^\alpha)_L, \varphi^\alpha) = (\vartheta_L, \varphi) = (\vartheta_L^g, \varphi^g) = (\vartheta_L^\alpha, \varphi^g),$$

and thus $\varphi^\alpha = \varphi^{gk}$ for some $k \in K$. It follows that $G = KT$. Since K/L is abelian, $K \cap T \trianglelefteq KT = G$ and thus either $K \cap T = L$ or $K \cap T = K$.

If $K \cap T = L$, then $\varphi^K = \vartheta$. Thus clearly $\mathbb{F}(\vartheta) \leq \mathbb{F}(\varphi)$. Let $\alpha \in \text{Gal}(\mathbb{F}(\varphi)/\mathbb{F}(\vartheta))$. Then $(\vartheta_L, \varphi^\alpha) = (\vartheta_L, \varphi) > 0$ and thus $\varphi^\alpha = \varphi^k$ for some $k \in K \cap T = L$, so that $\varphi^\alpha = \varphi$. It follows that $\text{Gal}(\mathbb{F}(\varphi)/\mathbb{F}(\vartheta)) = 1$, and thus $\mathbb{F}(\vartheta) = \mathbb{F}(\varphi)$, which is possibility (i) in situation (a).

Now suppose $K \cap T = K$, that is $T = G$ and φ is semi-invariant in G . Consider the inertia group K_φ . For $g \in G$, there is $\alpha_g \in \text{Aut } \mathbb{F}(\varphi)$ with $\varphi^g = \varphi^{\alpha_g}$, so that

$$K_\varphi^g = K_{\varphi^g} = K_{\varphi^{\alpha_g}} = K_\varphi.$$

It follows that $K_\varphi \trianglelefteq G$. Again, either $K_\varphi = K$ or $K_\varphi = L$.

If $K_\varphi = L$, then again $\varphi^K = \vartheta \in \text{Irr } K$, but now φ is semi-invariant in K . Since $\vartheta^k = \vartheta$ for $k \in K$, the homomorphism of Lemma 9.6 maps K/L into the Galois group $\text{Gal}(\mathbb{F}(\varphi)/\mathbb{F}(\vartheta))$. Conversely, for $\alpha \in \text{Gal}(\mathbb{F}(\varphi)/\mathbb{F}(\vartheta))$ we have $(\vartheta, \varphi^\alpha) = (\vartheta^\alpha, \varphi^\alpha) = 1$ and thus φ^α and φ are conjugate in K . It follows that the homomorphism of Lemma 9.6 is onto, and thus $K/L \cong \text{Gal}(\mathbb{F}(\varphi)/\mathbb{F}(\vartheta))$. This is situation (a)(ii).

Now assume $K_\varphi = K$, so that φ is invariant in K . Set

$$\Lambda = \{\lambda \in \text{Lin}(K/L) \mid \vartheta\lambda = \vartheta\} \quad \text{and} \quad U = \bigcap_{\lambda \in \Lambda} \text{Ker } \lambda.$$

We claim that $U \trianglelefteq G$. If $\vartheta\lambda = \vartheta$, then $\vartheta^\alpha\lambda = \vartheta^\alpha$ for field automorphisms α , as ϑ^α and ϑ have the same zeros. Let $g \in G$ and $\lambda \in \Lambda$. From the semi-invariance of ϑ it follows that there is $\alpha \in \text{Aut } \mathbb{F}(\vartheta)$ with $\vartheta^{\alpha g} = \vartheta$. Thus

$$\vartheta\lambda^g = \vartheta^{\alpha g}\lambda^g = (\vartheta^\alpha\lambda)^g = \vartheta^{\alpha g} = \vartheta.$$

Thus Λ is invariant in G , and it follows that $U \trianglelefteq G$. Thus either $U = K$ or $U = L$.

If $U = K$, then $\Lambda = \{1\}$ and thus the $\vartheta\lambda$ with $\lambda \in \text{Lin}(K/L)$ are $|K/L|$ different constituents of φ^K occurring with the same multiplicity, e , so that

$$|K/L|\varphi(1) = \varphi^K(1) = e|K/L|\vartheta(1) = e^2|K/L|\varphi(1),$$

and it follows $e = 1$ (situation (b)).

If $U = L$, then ϑ vanishes on $K \setminus L$, and thus φ is fully ramified in K (situation (c)). It is clear that then $\mathbb{F}(\vartheta) = \mathbb{F}(\varphi)$. \square

In situation (b), we clearly have $\mathbb{F}(\varphi) \leq \mathbb{F}(\vartheta)$, and

$$\{\vartheta^\alpha \mid \alpha \in \text{Gal}(\mathbb{F}(\vartheta)/\mathbb{F}(\varphi))\} \subseteq \text{Irr}(K \mid \varphi) = \{\vartheta\lambda \mid \lambda \in \text{Lin}(K/L)\}.$$

Thus $1 \leq |\mathbb{F}(\vartheta) : \mathbb{F}(\varphi)| \leq |K/L|$. In our intended application, we will have $\mathbb{F}(\vartheta) = \mathbb{F}(\varphi)$. Then the following result, probably well known, will be useful.

14.3. Proposition. *Let $H \leq G$ be finite groups, $K \trianglelefteq G$ with $G = HK$, and set $L = H \cap K$. Assume that $\vartheta \in \text{Irr } K$ is semi-invariant in G , that $\vartheta_L = \varphi \in \text{Irr } L$ and that $\mathbb{F}(\vartheta) = \mathbb{F}(\varphi)$ for some field $\mathbb{F} \subseteq \mathbb{C}$. Set $e = e_{(\vartheta, \mathbb{F})}$ and $f = e_{(\varphi, \mathbb{F})}$. Then*

$$\mathbb{F}Hf \ni a \mapsto ae \in \mathbb{F}Ge$$

is an isomorphism of H/L -graded algebras.

14.4. Corollary. *In the situation of the proposition, restriction defines a bijection $\text{Irr}(G \mid e) \rightarrow \text{Irr}(H \mid f)$ commuting with field automorphism over \mathbb{F} and preserving Schur indices over \mathbb{F} . (More precisely, the correspondence has Properties (a)–(i) of Proposition 10.4, with \mathbb{Q} replaced by \mathbb{F} .)*

Proof. The isomorphism $\mathbb{F}Hf \rightarrow \mathbb{F}Ge$ defines a map from $\text{Irr}(G \mid e)$ to $\text{Irr}(H \mid f)$, sending $\chi \in \text{Irr}(G \mid e)$ to χ^o with $\chi^o(h) = \chi(he) = \chi(h)$. It is clear that this commutes with field automorphisms. The part on the Schur indices follows since $\mathbb{F}He_{(\chi^o, \mathbb{F})} \cong \mathbb{F}Ge_{(\chi, \mathbb{F})}$. \square

Proof of Proposition 14.3. Let V be an absolutely irreducible module affording ϑ . Then $Ve_\varphi = V$ and $Ve_{\tilde{\varphi}} = 0$ for any other $\tilde{\varphi} \in \text{Irr}(L)$. It follows that $e_\varphi e_\vartheta = e_\vartheta$ and $e_{\tilde{\varphi}} e_\vartheta = 0$ for $\tilde{\varphi} \neq \varphi$. In particular, this holds for $\tilde{\varphi} = \varphi^\alpha$ when $1 \neq \alpha \in \text{Gal}(\mathbb{F}(\varphi)/\mathbb{F})$. Since we assume that $\mathbb{F}(\varphi) = \mathbb{F}(\vartheta)$, it follows that $e_{\varphi^\alpha} e_{\vartheta^\beta} = \delta_{\alpha, \beta} e_{\vartheta^\beta}$ for $\alpha, \beta \in \text{Gal}(\mathbb{F}(\varphi)/\mathbb{F})$. Thus $ef = e$.

Now $a \mapsto ae$ maps $\mathbb{F}Lf$ into $\mathbb{F}Ke$. Since $\mathbb{F}Lf$ is simple, the map is injective, and since $\mathbb{F}Lf$ and $\mathbb{F}Ke$ both have dimension $\varphi(1) = \vartheta(1)$ over its center, the map is an isomorphism. Finally, for $h \in H$, we get $\mathbb{F}Lf h \cdot e = \mathbb{F}Keh$. The proof follows. \square

We also need a standard fact about coprime action [12, Theorems 13.27, 13.28, 13.31 and Problem 13.10] or [10, Corollary 2.4 and Lemma 2.5].

14.5. Lemma. *Let A act on K and let $L \trianglelefteq K$ be A -invariant. Suppose $(|A|, |K/L|) = 1$ and $\mathbf{C}_{K/L}(A) = 1$. Then*

- (a) *If $\vartheta \in \text{Irr}_A K$ then ϑ_L has a unique A -invariant constituent.*
- (b) *If $\varphi \in \text{Irr}_A L$, then φ^K has a unique A -invariant constituent.*

The proof of the second assertion is relatively elementary if K/L is abelian [10, p. 2.5] and can be reduced to that case if K/L is solvable. We will only need this case. (In the case where K/L is not solvable, the proof depends on the Glauberman correspondence.) The first assertion is easy in any case.

Proof of Theorem 14.1. Suppose G is a counterexample with $|G/L|$ minimal.

As M/L acts coprimely and fixed point freely on K/L , it follows that above every $\varphi \in \text{Irr}_M L$, there lies a unique $\vartheta \in \text{Irr}_M K$, and conversely (see Lemma 14.5). Since this bijection is natural, it commutes with the action of H and with Galois action. In particular, $\mathbb{Q}(\vartheta) = \mathbb{Q}(\varphi)$ and $H_\vartheta = H_\varphi$.

Set $e_1 = e_{(\vartheta, \mathbb{Q})}$ and $f_1 = e_{(\varphi, \mathbb{Q})}$. Let U be the stabilizer of e_1 in G . Then $V = U \cap H$ is the stabilizer of f_1 in H . By Proposition 9.4 we have $\mathbb{Q}Ge \cong \mathbf{M}_{|G:U|}(\mathbb{Q}Ue_1)$ and $\mathbb{Q}Hf \cong \mathbf{M}_{|H:V|}(\mathbb{Q}Vf_1)$, and canonical character correspondences. If $U < G$, then induction applies and yields an isomorphism $\mathbb{Q}Ue_1 \cong \mathbf{M}_n(\mathbb{Q}Vf_1)$ as in the theorem and a canonical character correspondence. This yields

$$\mathbb{Q}Ge \cong \mathbf{M}_{|G:U|}(\mathbf{M}_n(\mathbb{Q}Vf_1)) \cong \mathbf{M}_n(\mathbf{M}_{|H:V|}(\mathbb{Q}Vf_1)) \cong \mathbf{M}_n(\mathbb{Q}Hf)$$

and canonical character correspondences. Thus the configuration can not be a minimal counterexample. It follows that $U = G$, that is, ϑ is semi-invariant in G .

In a counterexample, we must have $L < K$. Let $L < N \leq K$ with N/L a chief factor. There is a unique $\eta \in \text{Irr}_M N$ that lies above φ , and this η is a constituent of ϑ_N (Lemma 14.5). This η is also semi-invariant in H and has the same field of

values as φ and ϑ . Let $U = \mathbf{N}_G(MN)$. If $N < K$, then induction applies to yield isomorphisms $\mathbb{Q}Ge \cong \mathbf{M}_{n_1}(\mathbb{Q}Ui)$ and $\mathbb{Q}Ui \cong \mathbf{M}_{n_2}(\mathbb{Q}Hf)$ (where $i = e_{(\eta, \mathbb{Q})}$), and natural bijections between $\text{Irr}(G \mid \vartheta)$ and $\text{Irr}(U \mid \eta)$ with the required properties, and between $\text{Irr}(U \mid \eta)$ and $\text{Irr}(H \mid \varphi)$. It follows that $\mathbb{Q}Ge \cong \mathbf{M}_{n_1 n_2}(\mathbb{Q}Hf)$ with $n_1 n_2 = (\vartheta(1)/\eta(1)) \cdot (\eta(1)/\varphi(1)) = \vartheta(1)/\varphi(1)$, and that there is a natural bijection between $\text{Irr}(G \mid \vartheta)$ and $\text{Irr}(H \mid \varphi)$.

We may thus assume that K/L is a chief factor of G . Then, according to the “going down” result for semi-invariant characters (Proposition 14.2), one of three possibilities occurs.

First, suppose we are in Situation (a) of Proposition 14.2, so that $\varphi^K = \vartheta$. Here φ can not be semi-invariant in K , since this would imply $\mathbb{Q}(\vartheta) < \mathbb{Q}(\varphi)$ which is impossible. It follows that H is the inertia group of f . Then Proposition 9.4 applies and yields the result. (In this case, $n = |K/L|$.)

Now suppose that $\vartheta_L = \varphi \in \text{Irr } L$. As $\mathbb{Q}(\vartheta) = \mathbb{Q}(\varphi)$, Proposition 14.3 applies and yields the result. (In this case, $n = 1$.)

Thus we assume that φ is fully ramified in K . Then Theorem 10.3 and Proposition 10.4 (with $N = MK$) apply and yield the result. (In this case, $n = \sqrt{|K/L|}$.) \square

Note that oddness of $|K/L|$ was only applied in the last sentence of the proof (if solvability is assumed). Nevertheless the result is false for $|K/L|$ even.

Now assume that N is a finite group on which the group A acts. Suppose that $|N|$ and $|A|$ are relatively prime, and that $|N|$ is odd. As mentioned in the introduction, Isaacs used his results on fully ramified sections to construct a correspondence between $\text{Irr}_A N$ and $\text{Irr } \mathbf{C}_N(A)$. We call this the Isaacs correspondence. (Strictly speaking, we should call it the Isaacs part of the Glauberman-Isaacs correspondence.) We will need to recall the construction of the Isaacs correspondence in the proof of the next result.

14.6. Corollary. *Let N be a finite group of odd order, A a group such that $|N|$ and $|A|$ are relatively prime, and suppose the semidirect product AN is a normal subgroup of a finite group G . Set $C = \mathbf{C}_N(A)$ and $U = \mathbf{N}_G(A)$. Let $\chi \in \text{Irr}_A(N)$ and $\chi^* \in \text{Irr } C$ be its Isaacs correspondent. Set $i = (e_{(\chi, \mathbb{Q})})^G$ and $i^* = (e_{(\chi^*, \mathbb{Q})})^U$. Then $\mathbb{Q}Gi \cong \mathbf{M}_n(\mathbb{Q}Ui^*)$ as (U/C) -graded algebras, with $n = \chi(1)/\chi^*(1)$. There is a natural correspondence between $\text{Irr}(G \mid i)$ and $\text{Irr}(U \mid i^*)$ preserving Schur indices.*

14.7. Corollary. *The Isaacs correspondence preserves Schur indices.*

Proof. The isomorphism of Corollary 14.6 restricts to an isomorphism $\mathbb{Q}Ni \cong \mathbf{M}_n(\mathbb{Q}Ci^*)$. \square

In the situation of Corollary 14.6, observe that $G = NU$ by the Frattini argument, and that $C = N \cap U$ since $(|A|, |N|) = 1$. It follows that $G/N \cong U/C$, and it makes sense to compare the character sets above χ respective χ^* . G. Navarro [23] attributes to L. Puig the question if the Clifford extensions of G/N and U/C associated to χ and χ^* are isomorphic in this case. This has been answered in the affirmative by M. L. Lewis [21]. (To be exactly, he shows that the associated character triples are isomorphic, which is somewhat weaker.) Corollary 14.6 generalizes this result. If $|N|$ even, that is, we are in the situation of the Glauberman correspondence, the result is false. However, it is true if we work over \mathbb{C} instead of \mathbb{Q} [5, 25, 21]. If A is a p -group, then it is true over \mathbb{Q}_p , the p -adic numbers [27].

Proof of Corollary 14.6. Let $K = [N, A]$ and $L = K'$. Then $L, K \trianglelefteq G$. We may assume that $K > 1$ (otherwise $C = N$ and $G = U$). It follows that $L < K$. By results on coprime action, $\mathbf{C}_{K/L}(A) = 1$.

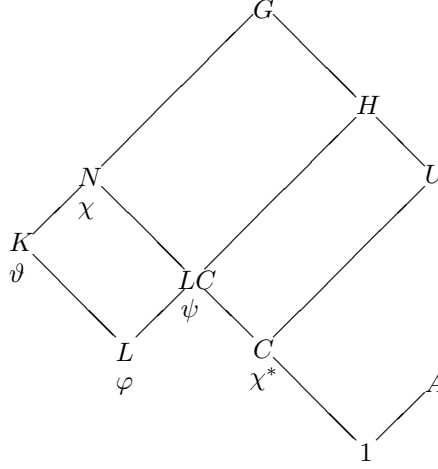


FIGURE 3. Corollary 14.6

There is an A -invariant constituent ϑ of χ_K [12, Theorem 13.27]. By Lemma 14.5, there is a unique A -invariant constituent φ of ϑ_L . Set $M = AL$ and $H = \mathbf{N}_G(M)$. Then $MK = AK \trianglelefteq G$, $\mathbf{C}_{K/L}(M) = 1$ and, by the Frattini argument, $H = \mathbf{N}_G(A)L = UL$ (see Figure 3).

Now Theorem 14.1 applies and yields an isomorphism $\kappa: \mathbb{Q}Ge \rightarrow \mathbf{M}_{n_1}(\mathbb{Q}Hf)$, where e and f are as in Theorem 14.1. The natural correspondence sends χ to an character $\psi \in \text{Irr}(LC)$. By the inductive definition of the Isaacs correspondence, ψ is the Isaacs correspondent of χ^* . Let j be the idempotent in $\mathbf{Z}(\mathbb{Q}LC)^H$ belonging to ψ . Then κ restricts to an isomorphism $\mathbb{Q}Gi \rightarrow \mathbf{M}_{n_1}(\mathbb{Q}Hj)$. Observe that $n_1 = \chi(1)/\psi(1)$. By induction, $\mathbb{Q}Hj \cong \mathbf{M}_{n_2}(\mathbb{Q}Ui^*)$ with $n_2 = \psi(1)/\chi^*(1)$. The result now follows. \square

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